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# Solitary Waves in Compressible Media

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### Abstract

Solitary waves in compressible media of finite depth and infinite depth are studied. The critical speeds are first obtained from the linearized equations and then confirmed by the results of the nonlinear theory. Explicit expressions for the solitary waves are established by a perturbation scheme applied to the nonlinear equations.

The case of a polytropic compressible medium of finite depth at rest in the state of equilibrium is studied in Part I. Solitary waves in compressible medium of infinite depth are investigated in Part II and Part III. The former concerns two isothermal layers at rest in the state of equilibrium separated by a contact surface; the latter, an isothermal layer with non-uniform velocity distribution at equilibrium. It is found that solitary waves vanish at certain values of characteristic parameters introduced in each case, and especially no solitary wave solution exists for an isothermal layer of infinite depth.



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## Part I. Polytropic Compressible Media

### 1. Introduction

The main purpose of this work is to extend Peters and Stoker's scheme [1] to the study of solitary waves in gravitating, polytropic or isothermal compressible media of infinite or finite depth. These are steady two-dimensional flows over a plane level bottom with a free surface which may or may not be at infinity. The solitary waves are waves of permanent type moving with constant velocity in the horizontal direction, and the vertical displacement of the stream lines has only a single crest or trough and tends to the equilibrium state at infinity. An interesting outline of the history and many physical aspects of the solitary wave problem have been given in [1]. As a supplementary note to the bibliography cited there we would like to mention that the problem of a solitary wave in an incompressible medium of non-uniform primary velocity distribution has been solved recently by Benjamin [2]. He used the vertical distance at equilibrium as one of the independent variables in place of the stream function. With an approach different from [2] we shall introduce the same independent variable to modify our scheme in order to study solitary waves in compressible media with non-uniform velocity distribution at equilibrium.



In Part I we shall consider the simplest case of a solitary wave in a one-layer polytropic compressible medium with a view to showing the general approach to the problem of this kind. By polytropic compressible medium we mean that for the compressible medium there exists a relation between pressure  $p$  and density  $\rho$ , i.e.  $p = \rho^n$  where  $n > 1$ . For  $n = 1$  we call the medium an isothermal compressible medium if the equation of state for a perfect gas is used. The solitary wave problem is first formulated in Section 2. In Section 3 the solution of the linearized equations predicts the value of the critical speed defined there, and in Section 4 the solitary wave near the critical speed is investigated by the nonlinear theory. The results are discussed in Section 5. It is interesting to note that the solitary wave solution does not exist for  $n = 1$  and  $n \cong 5.15$  under the present perturbation scheme. The former corresponds to an isothermal layer of infinite depth and the latter, a polytropic layer of finite depth. Furthermore, for  $n > 5.15$  ( $1 < n < 5.15$ ) the speed of the solitary wave is greater than (less than) the speed  $\sqrt{gh}$  where  $h$  is a characteristic length defined later, and  $g$  the gravitational constant, and the solitary wave is one of depression (of elevation).



In Part II solitary waves in two isothermal layers are studied. These two layers are separated by an interface, the so-called contact surface, across which pressure and velocity are continuous but density and temperature are subject to a discontinuity. In principle it is not difficult to extend the method to the cases of two or more than two layers in each of which  $n$  assumes different value; however, since more parameters must be introduced, the algebraic calculations will become prohibitive. For the case studied in Part II the upper layer at equilibrium is extended to infinity and at constant temperature  $T_2$ ; and its lower layer at equilibrium is of a finite height  $h$  and at temperature  $T_1$ . We always assume that  $\alpha \equiv \frac{T_2}{T_1} > 1$ .

Let  $r \equiv 1 + [\exp \frac{gh}{\tilde{p}_0/\tilde{\rho}_0} - 1]^{-1}$ , where  $\tilde{p}_0$ ,  $\tilde{\rho}_0$  are the

equilibrium pressure and density at the plane bottom. Then for a given set of  $\alpha$  and  $r$  there exist two critical speeds. Corresponding to each critical speed the domain  $\alpha > 1$ ,  $r > 1$  in the  $\alpha, r$ -plane is divided into several subdomains, in each of which the solitary wave may be a wave of elevation or depression, and its speed may be greater or less than the critical speed. Along the boundary of these subdomains the solitary wave solution will not exist under the present scheme.

In Part III we shall investigate solitary waves in an isothermal compressible medium with non-uniform velocity



distribution in the equilibrium state. It is worth-while to mention that the solitary wave solution will not exist as the velocity becomes uniform. This confirms what we have shown in Part I.

Finally, we would like to remark that since compressibility plays an important role in the study of the atmosphere, these results obtained, if relevant, may explain certain geophysical phenomena related to gravitational waves in compressible media.

## 2. Formulation of the Problem

We assume that a body of polytropic or isothermal compressible medium is supported by a plane rigid bottom and has a free surface on which the pressure  $p$  is zero and there are no geometric constraints. A cross section of the medium in the equilibrium state is a horizontal strip of finite or infinite depth. Let us assume that a two-dimensional wave of permanent type which moves to the left with velocity  $c$  has been created by some disturbance in the medium initially at rest. We choose a coordinate system moving with the wave such that the  $x$ -axis coincides with the bottom and the  $y$ -axis passes through the crest for a wave of elevation and the trough for a wave of depression (Fig. 1). As observed from the coordinate system the wave is stationary and the velocity



of the medium moving to the right at infinity is  $c$ .

The steady state equations governing the two-dimensional motion of a polytropic or isothermal compressible medium under a gravitational field are the equation of continuity:

$$(2.1) \quad \frac{\partial(\tilde{\rho}\tilde{u})}{\partial x} + \frac{\partial(\tilde{\rho}\tilde{v})}{\partial y} = 0 ,$$

the equations of motion:

$$(2.2) \quad \begin{aligned} \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} &= - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial x} , \\ \tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial y} &= -g - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial y} , \end{aligned}$$

the specifying equation:

$$(2.3) \quad \tilde{p}/\tilde{p}_0 = (\tilde{\rho}/\tilde{\rho}_0)^n \quad n \geq 1 ,$$

and the equation of state:

$$f(\tilde{p}, \tilde{\rho}, \tilde{T}) = 0 ,$$

where  $\tilde{u}(x,y)$  ,  $\tilde{v}(x,y)$  are respectively the horizontal and the vertical velocity component,  $\tilde{p}(x,y)$  ,  $\tilde{\rho}(x,y)$  and  $\tilde{T}(x,y)$  are respectively the pressure, density, and temperature, and  $g$  is the gravitational constant. In what follows we shall



always assume that the equation of state takes the form

$$(2.4) \quad \tilde{p} = R \tilde{\rho} \tilde{T}$$

as the one for a perfect gas where  $R$  is the gas constant.

For  $n = 1$  the flow is isothermal, and for an isentropic flow of a perfect gas  $n = \gamma = 1.4$ . At the bottom and on the free surface two conditions are imposed:

At the bottom  $y = 0$  :

$$(2.5) \quad v = 0 ;$$

on the free surface  $\tilde{S}(x_S, y_S) = 0$  ,

$$(2.6) \quad \tilde{u} \frac{\partial \tilde{S}}{\partial x} + \tilde{v} \frac{\partial \tilde{S}}{\partial y} = 0 .$$

Let us first consider the medium in the equilibrium state moving with constant velocity  $\tilde{u}_\infty = c$ . Hereafter we use the subscript  $\infty$  to denote the variables in the equilibrium state.

From (2.1)  $\frac{\partial \tilde{\rho}_\infty}{\partial x} = 0$  and  $\tilde{\rho}_\infty = \tilde{\rho}_\infty(y)$ .

(2.2) only gives  $\tilde{p}_\infty = \tilde{p}_\infty(y)$  which is consistent with (2.3).

Finally from (2.2) to (2.4) it is obtained that for  $n > 1$

$$(2.7) \quad \frac{\tilde{\rho}_\infty}{\tilde{\rho}_0} = \left(1 - \frac{n-1}{n} \frac{y}{h}\right)^{\frac{1}{n-1}},$$

$$\frac{\tilde{p}_\infty}{\tilde{p}_0} = \left(1 - \frac{n-1}{n} \frac{y}{h}\right)^{\frac{n}{n-1}},$$



$$\frac{z_{H_0}}{z_{H_0}} = (1 - \frac{n-1}{n} \frac{y}{h}) ;$$

and for  $n = 1$  ,

$$\frac{z_{\infty}}{z_0} = \exp(- \frac{y}{h}) ,$$

$$(2.8) \quad \frac{z_{\infty}}{z_0} = \exp(- \frac{y}{h}) ,$$

$$\tilde{T}_{\infty} = \tilde{T}_0 ,$$

where  $\tilde{p}_0$  ,  $\tilde{\rho}_0$  and  $\tilde{T}_0$  are respectively the pressure, density, and temperature at  $y = 0$  , and  $h = \frac{\tilde{p}_0}{g\tilde{\rho}_0}$  . Suppose that

$\tilde{p}_0$  ,  $\tilde{\rho}_0$  are always positive finite. Then it is seen from (2.7) that for a polytropic medium of  $n > 1$  ,  $\tilde{p}_{\infty}$  ,  $\tilde{\rho}_{\infty}$  and  $\tilde{T}_{\infty}$  are equal to zero at  $y = \frac{n}{n-1} h$  and the only possible solution for  $y > \frac{n}{n-1} h$  is  $\tilde{p}_{\infty}$  ,  $\tilde{\rho}_{\infty}$  and  $\tilde{T}_{\infty} = 0$  . From (2.8) we can also observe that for an isothermal layer  $\tilde{p}_{\infty}$  ,  $\tilde{\rho}_{\infty} \rightarrow 0$  as  $y \rightarrow \infty$  . Therefore, in the equilibrium state a polytropic layer is of finite depth  $\frac{n}{n-1} h$  while an isothermal layer is of infinite depth.

Now we introduce a stream function  $\tilde{\psi}(x,y)$  such that

$$(2.9) \quad \frac{\partial \tilde{\psi}}{\partial y} = \tilde{\rho} \tilde{u} , \quad \frac{\partial \tilde{\psi}}{\partial x} = - \tilde{\rho} \tilde{v} .$$



From (2.5) and (2.6) it is seen that both the bottom and the free surface are stream lines, i.e.  $\tilde{\psi} = \text{const}$  along  $y = 0$  and  $\tilde{S}(x_S, y_S) = 0$ . We define  $\tilde{\psi}(x, 0) \equiv 0$ . The constant value of  $\tilde{\psi}(x_S, y_S)$  along  $\tilde{S}(x_S, y_S) = 0$ ,

$$(2.10) \quad \tilde{\psi}(x_S, y_S) = \int_{(x_S, 0)}^{(x_S, y_S)} (-\tilde{\rho} \tilde{v} dx + \tilde{\rho} \tilde{u} dy),$$

gives the mass flux across any vertical plane from bottom to the height  $y_S$  per unit breadth. In the equilibrium state  $\tilde{u}_\infty \equiv c$ , for  $n > 1$ ,

$$\tilde{\psi}(x_S, y_S) = \int_0^{\frac{n}{n-1} h} \tilde{\rho}_\infty \tilde{u}_\infty dy = \tilde{\rho}_0 c h,$$

and for  $n = 1$ ,

$$\tilde{\psi}(x_S, y_S) = \int_0^\infty \tilde{\rho}_\infty \tilde{u}_\infty dy = \tilde{\rho}_0 c h.$$

Hence along  $\tilde{S}(x_S, y_S) = 0$ ,

$$(2.11) \quad \tilde{\psi} \equiv \tilde{\rho}_0 c h.$$

The totality of stream lines is given implicitly by

$$\tilde{\psi}(x, y) = \gamma = \text{const}.$$

It is assumed that to each value of  $\gamma$  such that  $0 \leq \gamma < \tilde{\rho}_0 c h$  there exists a unique solution of  $y$  for the above equation, i.e.

$$(2.12) \quad y = \bar{f}(x, \gamma).$$



The bar notation is used hereafter to indicate a function of  $x, \gamma$ . From (2.9) and (2.12)

$$(2.13) \quad \begin{aligned} \bar{f}_\gamma &= \frac{1}{\tilde{\psi}_y} = \frac{1}{\bar{\rho} \bar{u}}, \\ \bar{f}_x &= -\tilde{\psi}_x \bar{f}_\gamma = \frac{\bar{v}}{\bar{u}}. \end{aligned}$$

Now for any function  $\tilde{\Phi}(x, y) = \bar{\Phi}(x, \gamma)$ ,

$$(2.14) \quad \begin{aligned} \frac{\partial \tilde{\Phi}}{\partial x} &= \frac{\partial \bar{\Phi}}{\partial x} + \frac{\partial \bar{\Phi}}{\partial \gamma} \frac{\partial \gamma}{\partial x} = \frac{\partial \bar{\Phi}}{\partial x} - \bar{\rho} \bar{v} \frac{\partial \bar{\Phi}}{\partial \gamma}, \\ \frac{\partial \tilde{\Phi}}{\partial y} &= \frac{\partial \bar{\Phi}}{\partial x} \frac{\partial \gamma}{\partial x} = \bar{\rho} \bar{u} \frac{\partial \bar{\Phi}}{\partial \gamma}. \end{aligned}$$

By using  $\bar{f}$ ,  $\bar{u}$ ,  $\bar{\rho}$ , and  $\bar{p}$  as dependent variables, the equations of motion can be transformed to  $x, \gamma$ -plane with the help of (2.13) to (2.14). From (2.2), (2.3), (2.13) and

(2.14) we have, for  $0 < \gamma < \tilde{\rho}_0 ch$ ,  $-\infty < x < +\infty$ ,

$$\bar{u}(\bar{u}_x - \bar{\rho} \bar{v} \bar{u}_\gamma) + \bar{v} \bar{\rho} \bar{u} \bar{u}_\gamma = -\frac{1}{\bar{\rho}}(\bar{p}_x - \bar{\rho} \bar{v} \bar{p}_\gamma),$$

$$\text{i.e.} \quad \bar{u}_x = -\bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma;$$

(2.15)

$$\text{and} \quad \bar{u}(\bar{v}_x - \bar{\rho} \bar{v} \bar{v}_\gamma) + \bar{v} \bar{\rho} \bar{u} \bar{v}_\gamma = -g - \frac{1}{\bar{\rho}}(\bar{\rho} \bar{u} \bar{p}_\gamma),$$

$$\text{i.e.} \quad \bar{u} \bar{f}_{xx} + \bar{u}_x \bar{f}_x = -\bar{\rho} g \bar{f}_\gamma - \bar{p}_\gamma.$$



Together with (2.15), we have

$$\bar{\rho} \bar{u} \bar{f}_\gamma = 1$$

in place of the equation of continuity, and the specifying equation

$$\frac{\bar{p}}{\tilde{p}_0} = \left( \frac{\bar{\rho}}{\tilde{\rho}_0} \right)^n \quad n \geq 1$$

to give a relation between  $\bar{p}$  and  $\bar{\rho}$ . The boundary conditions are:

$$\bar{f}(x, 0) = 0 ,$$

(2.16)

$$\bar{p}(x, \tilde{\rho}_0 \text{ch}) = 0 .$$

However, the above transformation is only a formal one. It is seen from (2.13) that as  $\bar{\rho} \rightarrow 0$ ,  $\bar{f}_\gamma \rightarrow \infty$ , and the transformation breaks down. In fact, the free surface in the  $x, y$ -plane is the so-called vacuum line and the image in the  $x, \gamma$ -plane becomes a branch line [3]. All the values of the variables at the branch line must be defined as the limiting values of these variables when  $\gamma \uparrow \tilde{\rho}_0 \text{ch}$ .

In the following, we introduce the dimensionless variables

$$\xi = \frac{x}{h} , \quad v = \frac{\bar{v}}{c} , \quad u = \frac{\bar{u}}{c} ,$$



$$\eta = \frac{\gamma}{\tilde{\rho}_0^{\text{ch}}} , \quad p = \frac{\bar{p}}{\tilde{\rho}_0 c^2} , \quad \rho = \frac{\bar{p}}{\tilde{\rho}_0}$$

$$f = \frac{\bar{f}}{h} , \quad \lambda = \frac{gh}{c^2} ,$$

where  $h = \frac{\tilde{p}_0}{\tilde{\rho}_0 g}$ . Then the equations (2.3), (2.13), (2.15),

and (2.16) become, for  $0 < \eta < 1$  ,  $-\infty < \xi < +\infty$  ,

$$u_\xi = -f_\eta p_\xi + f_\xi p_\eta ,$$

$$(2.17) \quad u f_{\xi\xi} + u_\xi f_\xi = -\lambda \rho f_\eta - p_\eta ,$$

$$\rho u f_\eta = 1 ,$$

$$p = \lambda \rho^n ,$$

$$f(\xi, 0) = 0 , \quad p(\xi, 1) = 0 .$$

### 3. Linear Theory. Critical Speed

The solution of (2.17) for the equilibrium state  $u \equiv 1$  is found as follows:

$$p_0 = \lambda(1-\eta) , \quad \rho_0 = (1-\eta)^{\frac{1}{n}}$$



$$f_0 = \frac{n}{n-1} [1 - (1-\eta)^{\frac{n-1}{n}}] , \quad \text{for } n > 1 ,$$

$$= -\log(1-\eta) , \quad \text{for } n = 1 .$$

We assume that the wave motion is a small disturbance superposed on the equilibrium state, and let

$$p = \lambda(1-\eta) + p^* , \quad \rho = (1-\eta)^{\frac{1}{n}} + \rho^* ,$$

$$f = \frac{n}{n-1} [1 - (1-\eta)^{\frac{n-1}{n}}] + f^* , \quad \text{for } n > 1 ,$$

$$= -\log(1-\eta) + f^* , \quad \text{for } n = 1 ,$$

$$u = 1 + u^* .$$

Here we may suppose that all the starred quantities are uniformly small for  $0 \leq \eta \leq 1$  ,  $-\infty < \xi < +\infty$  ; however it is unlikely that the same would hold for the derivatives of  $f^*$  since  $f_\eta$  has a singularity at  $\eta = 1$  . For the time being let us substitute these quantities in (2.17) and proceed formally to neglect all the terms containing the second order products of the starred quantities and their derivatives. In the Appendix we shall examine whether our linearizing procedure is actually legitimate. The linearized equations are found to be

$$u_\xi^* = - (1-\eta)^{-\frac{1}{n}} p_\xi^* - \lambda f_\xi^* ,$$



$$\begin{aligned}
 (3.1) \quad f_{\xi\xi}^* &= -\lambda(1-\eta)^{\frac{1}{n}} f_{\xi}^* - \lambda(1-\eta)^{-\frac{1}{n}} \rho^* - p_{\eta}^* , \\
 u^* &= - (1-\eta)^{-\frac{1}{n}} \rho^* - (1-\eta)^{\frac{1}{n}} f_{\eta}^* , \\
 \rho^* &= \frac{1}{n\lambda}(1-\eta)^{\frac{1}{n}-1} p^* ,
 \end{aligned}$$

subject to the boundary conditions

$$f^*(\xi, 0) = 0 , \quad p^*(\xi, 1) = 0 .$$

From (3.1) it is found that

$$\begin{aligned}
 &[(1-\eta)^{-\frac{1}{n}} - \frac{1}{n\lambda}(1-\eta)^{-1}]^2 f_{\xi\xi\xi}^* + [1 - \frac{1}{n\lambda}(1-\eta)^{\frac{1}{n}-1}] f_{\xi\eta\eta}^* \\
 &+ [-\frac{2}{n}(1-\eta)^{-1} + \frac{1}{\lambda}(\frac{1}{n} + \frac{1}{n^2})(1-\eta)^{\frac{1}{n}-2}] f_{\xi\eta}^* \\
 (3.2) \quad &+ [(\frac{1}{n^2} - \frac{1}{n})(1-\eta)^{-2}] f_{\xi}^* = 0 ,
 \end{aligned}$$

$$p_{\xi}^* = [(1-\eta)^{-\frac{1}{n}} - \frac{1}{n\lambda}(1-\eta)^{-1}]^{-1} [(1-\eta)^{\frac{1}{n}} f_{\xi\eta}^* - \lambda f_{\xi}^*] .$$

Let

$$f_{\xi}^* = F(\eta) G(\xi) ,$$

then from (3.2)

$$G_{\xi\xi} + v^2 G = 0 ,$$



$$\begin{aligned}
& \left[ 1 - \frac{1}{n\lambda} (1-\eta)^{\frac{1}{n}-1} \right] F_{\eta\eta} + \left[ -\frac{2}{n} (1-\eta)^{-1} + \lambda^{-1} \left( \frac{1}{n} + \frac{1}{n^2} \right) (1-\eta)^{\frac{1}{n}-2} \right] F_{\eta} \\
& + \left( \frac{1}{n^2} - \frac{1}{n} \right) (1-\eta)^{-2} F = v^2 \left[ (1-\eta)^{-\frac{1}{n}} - \frac{1}{n\lambda} (1-\eta)^{-1} \right]^2 F,
\end{aligned}$$

(3.3)

where  $F$  must satisfy the boundary conditions

$$F(0) = 0,$$

$$\lim_{\eta \rightarrow 1^-} (1-\eta) \left[ (1-\eta)^{1-\frac{1}{n}} - \frac{1}{n\lambda} \right]^{-1} \left[ (1-\eta)^{\frac{1}{n}} F_{\eta} - \lambda F \right] = 0.$$

The solution for  $G(\xi)$  is seen as

$$G(\xi) = A \cos(v\xi + B),$$

where  $A, B$  are arbitrary constants. However, since we are only interested in finding the value of the critical speed  $\ell$  defined as

$$\ell = \lim_{v \rightarrow 0} \lambda(v), \quad (1)$$

for simplicity, the following asymptotic method is used to solve the equation for  $F$  in (3.3) for  $n > 1$ , while a general discussion of the solution will be deferred to the Appendix. Let us suppose that, for small values of  $v$ ,

---

(1) For a discussion of the definition of critical speeds, c.f. [1].



$$\lambda = \ell + v^2 \lambda_1 + v^4 \lambda_2 + \dots ,$$

$$F = F_0(\eta) + v^2 F_1(\eta) + v^4 F_2(\eta) + \dots .$$

The equation governing  $F_0(\eta)$  is given by

$$\begin{aligned} & [(n\ell - (1-\eta)^{\frac{1}{n}-1}] F_{0\eta\eta} + [-2\ell(1-\eta)^{-1} + (1 + \frac{1}{n})(1-\eta)^{\frac{1}{n}-2}] F_{0\eta} \\ & + \ell(\frac{1}{n} - 1)(1-\eta)^{-2} F = 0 , \end{aligned}$$

subject to the boundary conditions

$$F_0(0) = 0 ,$$

$$\lim_{\eta \rightarrow 1^-} n\ell(1-\eta)[n\ell(1-\eta)^{1-\frac{1}{n}} - 1][(1-\eta)^{\frac{1}{n}} F_{0\eta} - \ell F_0] = 0 .$$

The solution of (3.4) is found as

$$F_0 = C[1 - \ell(1-\eta)^{\frac{n-1}{n}}] + D(1-\eta)^{-\frac{1}{n-1}} ,$$

where  $C$  and  $D$  are arbitrary constants. By the condition at  $\eta = 1$ , we have  $D = 0$ ; and if we assume  $C \neq 0$ , i.e. the motion is other than a parallel flow, then by  $F_0(0) = 0$  the critical speed is

$$\ell = 1 .$$

It is not difficult to find the higher order approximations



for the solution of  $F$ , however, since we have determined the value of the critical speed we will not proceed any farther.

For  $n = 1$ , the equation for  $F$  in (3.3) becomes

$$(1-\lambda^{-1})F_{\eta\eta} - 2(1-\eta)^{-1}(1-\lambda^{-1})F_{\eta} = v^2(1-\eta)^{-2}(1-\lambda^{-1})^2 F.$$

(1) Suppose  $\lambda \neq 1$ . We find that, by  $F(0) = 0$ ,

$$F = C_1[(1-\eta)^{m_1} - (1-\eta)^{m_2}],$$

where  $C_1$  is an arbitrary constant, and

$$m_1 = \frac{1}{2}[-1 + (1 + 4v^2(1-\lambda^{-1}))^{1/2}],$$

$$m_2 = \frac{1}{2}[-1 - (1 + 4v^2(1-\lambda^{-1}))^{1/2}].$$

This solution is unbounded at  $\eta = 1$ . Therefore, in this case either linear theory fails or we must set  $C_1 = 0$  and  $F \equiv 0$ .

(2) Suppose  $\lambda = 1$ . We obtain from the equation for  $p_{\xi}^*$  in (3.2) that

$$(1-\eta)F_{\eta} - F = 0,$$

and  $F \equiv 0$  if  $F(0) = 0$ . Therefore, we conclude that for the case of an isothermal layer of infinite depth either the linear theory fails or the solution must be identically equal to zero.



#### 4. Nonlinear Theory. Solitary Wave Solution

Let us assume that a solitary wave moves with a speed such that  $\lambda = \frac{gh}{c^2}$  is near some positive value  $\ell$ , which is to be determined later. The equations (2.17) can be written in the form, for  $0 < \eta < 1$ ,  $-\infty < \xi < +\infty$ ,

$$u_{\xi} = -f_{\eta}p_{\xi} + f_{\xi}p_{\eta},$$

$$uf_{\xi\xi} + u_{\xi}f_{\xi} = (\ell - \lambda)\rho f_{\eta} - \ell\rho f_{\eta} - p_{\eta},$$

$$\rho u f_{\eta} = 1$$

$$p = -(\ell - \lambda)\rho^n + \ell\rho^n, \quad n \geq 1.$$

Let

$$\varepsilon = \ell - \lambda, \quad \sigma = \sqrt{\varepsilon} \xi,$$

the above equations become

$$u_{\sigma} = -f_{\eta}p_{\sigma} + f_{\sigma}p_{\eta},$$

$$\varepsilon(uf_{\sigma\sigma} + u_{\sigma}f_{\sigma}) = \varepsilon\rho f_{\eta} - \ell\rho f_{\eta} - p_{\eta}$$

(4.1)

$$\rho u f_{\eta} = 1,$$

$$p = -\varepsilon\rho^n + \ell\rho^n, \quad n \geq 1$$



together with the boundary conditions

$$f(\sigma, 0) = 0, \quad p(\sigma, 1) = 0.$$

We assume that all the dependent variables can be expanded in integral powers of  $\varepsilon$ , i.e.

$$(4.2) \quad \phi(\sigma, \eta, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \phi_k(\sigma, \eta)$$

where  $\phi(\sigma, \eta, \varepsilon)$  stands for  $f$ ,  $u$ ,  $p$ , and  $\rho$ . Substitution of (4.2) in (4.1) yields a sequence of equations for  $f_k$ ,  $u_k$ ,  $p_k$ , and  $\rho_k$ . The equations for the zero-th order approximation are, for  $0 < \eta < 1$ ,  $-\infty < \sigma < +\infty$ ,

$$u_{0\sigma} = -p_{0\sigma} f_{0\eta} + f_{0\sigma} p_{0\eta},$$

$$0 = -\ell \rho_0 f_{0\eta} - p_{0\eta},$$

(4.3)

$$\rho_0 u_0 f_{0\eta} = 1,$$

$$p_0 = \ell \rho_0^n, \quad n \geq 1,$$

subject to the boundary conditions

$$f_0(\sigma, 0) = 0, \quad p_0(\sigma, 1) = 0.$$

We assume that  $u_0 \equiv 1$  and the solutions of the above equations are



$$p_0 = \ell(1-\eta) , \quad \rho_0 = (1-\eta)^{\frac{1}{n}} , \quad \text{for } n \geq 1 ,$$

$$f_0 = \frac{n}{n-1} [1 - (1-\eta)^{\frac{n-1}{n}}] , \quad \text{for } n > 1 ,$$

(4.4)

$$= -\log(1-\eta) , \quad \text{for } n = 1 ,$$

$$u_0 = 1 .$$

The equations for the first order approximation are,  
for  $0 < \eta < 1$  ,  $-\infty < \sigma < +\infty$  ,

$$u_{1\sigma} = -f_{0\eta} p_{1\sigma} + f_{1\sigma} p_{0\eta} ,$$

$$0 = \rho_0 f_{0\eta} - \ell(\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - p_{1\eta} ,$$

(4.5)

$$\rho_1 u_0 f_{0\eta} + \rho_0 u_1 f_{0\eta} + \rho_0 u_0 f_{1\eta} = 0 ,$$

$$p_1 = \ell n \rho_1 \rho_0^{n-1} - \rho_0^n ,$$

subject to

$$f_1(\sigma, 0) = 0 , \quad p_1(\sigma, 1) = 0 .$$

Elimination of  $u_1$  ,  $f_1$  , and  $\rho_1$  from (4.5) yields  
a simple equation

$$p_{1\sigma\eta\eta} = 0$$

for  $p_1$  . The solution of  $p_1$  satisfying  $p_1(\sigma, 1) = 0$  is



$$(4.6) \quad p_{1\sigma} = a_1'(\sigma)(1-\eta)$$

where  $a_1'(\sigma)$  is arbitrary. By integration of (4.6) with respect to  $\sigma$  we have

$$p_1 = a_1(\sigma)(1-\eta) + b_1(\eta) .$$

Since we have assumed that the wave motion tends to its equilibrium state at infinity, i.e.  $a_1(\sigma) \rightarrow 0$  as  $\sigma \rightarrow -\infty$ ,  $b_1(\eta) = 0$ . Thus

$$p_1 = a_1(\sigma)(1-\eta) ,$$

where we assume that  $a_1(\sigma) \neq 0$ . It is also obtained from (4.4) and (4.5) that

$$\begin{aligned} f_{1\sigma} &= (p_{0\eta})^{-1} (\ell^{-1} p_{1\eta\sigma} + f_{0\sigma} p_{1\sigma}) \\ &= \ell^{-2} a_1'(\sigma) [1 - \ell(1-\eta)^{\frac{n-1}{n}}] , \quad \text{for } n \geq 1 ; \end{aligned}$$

and by the equilibrium condition at infinity we have

$$f_1 = \ell^{-2} a_1(\sigma) [1 - \ell(1-\eta)^{\frac{n-1}{n}}] , \quad \text{for } n \geq 1 .$$

Since  $f_1(\sigma, 0) = 0$ , it follows that

$$\ell = 1 , \quad \text{for } n > 1 ,$$

which confirms the value for the critical speed we have obtained by the linearized equations, and also the assumption



of positive finiteness of  $\ell$ . For  $n = 1$ , we have

$$f_1(\sigma, \eta) = 0.$$

This shows that for an isothermal layer of infinite depth the medium always remains in the state of equilibrium, i.e. at rest, as already discussed in the linear theory. The solution for the first order approximation is summarized as follows: for  $n > 1$ ,

$$(4.7) \quad \begin{aligned} p_1 &= a_1(\sigma)(1-\eta), & \rho_1 &= \frac{1}{n}[a_1(\sigma) + 1](1-\eta)^{\frac{1}{n}}, \\ f_1 &= a_1(\sigma)[1 - (1-\eta)^{\frac{n-1}{n}}], & u_1 &= -a_1(\sigma) - 1. \end{aligned}$$

The equations for the second order approximation are, for  $0 < \eta < 1$ ,  $-\infty < \sigma < +\infty$ ,

$$(4.8) \quad \begin{aligned} u_{2\sigma} &= -(p_{2\sigma}f_{0\eta} + p_{1\sigma}f_{1\eta}) + (f_{2\sigma}p_{0\eta} + f_{1\sigma}p_{1\eta}), \\ f_{1\sigma\sigma} &= (\rho_0f_{1\eta} + \rho_1f_{0\eta}) - (\rho_0f_{2\eta} + \rho_1f_{1\eta} + \rho_2f_{0\eta}) - p_{2\eta}, \\ \rho_0u_{2\eta}f_{0\eta} + \rho_0u_{0\eta}f_{2\eta} + \rho_2u_{0\eta}f_{0\eta} + \rho_0u_{1\eta}f_{1\eta} + \rho_1u_{0\eta}f_{1\eta} + \rho_1u_{1\eta}f_{0\eta} &= 0, \\ \rho_2 &= \frac{1}{n}p_0^{\frac{1-n}{n}}[p_2 + p_1 + p_0 - \frac{n(n-1)}{2}\rho_1^2\rho_0^{n-2}] \end{aligned}$$

subject to the boundary conditions

$$f_2(\sigma, 0) = 0, \quad p_2(\sigma, 1) = 0.$$



By elimination of  $u_2$ ,  $f_2$ , and  $\rho_2$  from the above equations we obtain

$$(4.9) \quad p_{2\sigma\eta\eta} = g_4(\sigma, \eta)$$

where

$$\begin{aligned} g_4(\sigma, \eta) = & g_{1\eta} - \rho_0^{-1} p_{0\eta} g_{2\sigma} - g_{2\sigma\eta} + p_{0\eta} \rho_0^{-1} g_{3\sigma} \\ & - \rho_0^{-1} p_{0\eta} f_{0\eta} \left\{ \frac{1}{n} \rho_0^{1-n} \left[ p_1 - \frac{n(n-1)}{2} \rho_1^2 \rho_0^{n-2} \right] \right\}_{\sigma} \end{aligned}$$

$$(4.10) \quad g_1(\sigma, \eta) = - p_{1\sigma} f_{1\eta} + f_{1\sigma} p_{1\eta},$$

$$g_2(\sigma, \eta) = f_{1\sigma\sigma} + u_1 + u_1^2,$$

$$g_3(\sigma, \eta) = - \rho_1 f_{1\eta} + u_1^2.$$

Since  $g_4(\sigma, \eta)$  is a known function of  $\sigma$  and  $\eta$ , let

$$G_4(\sigma, \eta) = \int_1^{\eta} g_4(\sigma, \eta') d\eta'$$

then the solution for  $p_{2\sigma}$  is, with  $a_2'(\sigma)$  arbitrary,

$$(4.11) \quad p_{2\sigma} = a_2'(\sigma)(1-\eta) + \int_1^{\eta} G_4(\sigma, \eta') d\eta'.$$

It is also obtained from (4.8) and (4.11) that



$$\begin{aligned}
f_{2\sigma} &= (p_{0\eta})^{-1} (p_{2\sigma\eta} + g_{2\sigma} + p_{2\sigma} f_{0\eta} - g_1) \\
&= (p_{0\eta})^{-1} (G_4(\sigma, \eta) + g_{2\sigma}(\sigma, \eta) + f_{0\eta} \int_1^\eta G_4(\sigma, \eta') d\eta' - g_1(\sigma, \eta)).
\end{aligned}$$

Since  $f_{2\sigma}(\sigma, 0) = 0$ , we have

$$(4.12) \quad G_4(\sigma, 0) + g_{2\sigma}(\sigma, 0) - \int_0^1 G_4(\sigma, \eta') d\eta' - g_1(\sigma, 0) = 0.$$

Making use of (4.4) and (4.7), we obtain from (4.12) by some involved but straight forward computation the following equation:

$$m_0 a_1'''(\sigma) + m_1 a_1'(\sigma) a_1(\sigma) + m_2 a_1'(\sigma) = 0.$$

where

$$m_0 = \frac{-2n^3 + 13n^2 - 15n + 5}{2(n-1)(2n-1)},$$

$$m_1 = \frac{3n-1}{n},$$

$$m_2 = \frac{3n-2}{2n-1}.$$

For reasons given in [1], we impose the conditions

$$a_1'(-\infty) = a_1'''(-\infty) = 0, \quad a_1'(0) = 0,$$

and the solution for  $a_1(\sigma)$  is

$$a_1(\sigma) = -\frac{3m_2}{m_1} \operatorname{sech}^2 \frac{1}{2} \sigma \sqrt{-\frac{m_2}{m_0}}.$$



If the results we have found can give an accurate approximation to the solitary wave in a polytropic compressible medium, then by returning to the  $x$ - $y$  variables, we have

(4.9)

$$\bar{p} \approx \tilde{\rho}_0 c^2 (1-\eta) \left[ 1 - (1-\lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)} \right],$$

$$\bar{\rho} \approx \tilde{\rho}_0 (1-\eta)^{\frac{1}{n}} \left[ 1 - (1-\lambda) \frac{1}{n} \left( 1 - \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)} \right) \right],$$

$$\bar{T} \approx \frac{1}{R} c^2 (1-\eta)^{1-\frac{1}{n}} \left[ 1 + (1-\lambda) \left( \frac{1}{n} - \left( 1 + \frac{1}{n} \right) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)} \right) \right],$$

$$\bar{f} \approx h \left[ 1 - (1-\eta)^{\frac{n-1}{n}} \right] \left[ \frac{n}{n-1} - (1-\lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)} \right],$$

$$\bar{u} \approx \frac{gh}{c} + c(1-\lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)},$$

$$\bar{v} \approx \frac{gh}{c} \left[ 1 - (1-\eta)^{\frac{n-1}{n}} \right] (\lambda-1) \left( \frac{\lambda-1}{m_0} \right)^{1/2} \frac{3m_2}{m_1}^{3/2}$$

$$\operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)} \tanh \frac{x}{2h} \sqrt{\frac{m_2}{m_0} (\lambda-1)},$$

where  $\eta = \frac{\gamma}{h}$ ,  $\lambda = \frac{gh}{c^2}$ ,  $h = \frac{\tilde{p}_0}{g\tilde{\rho}_0}$ . In terms of the vertical



distance,  $\xi$  from the bottom at  $x = -\infty$ ,

$$\gamma = \int_0^\xi \tilde{\rho}_\infty \tilde{u}_\infty dy = \tilde{\rho}_0 ch \left[ 1 - \left( 1 - \frac{n-1}{n} \frac{\xi}{h} \right)^{\frac{n-1}{n}} \right].$$

### 5. Properties of the Solitary Wave

We have shown that

$$\begin{aligned} a_1(\sigma) &= - \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{m_2}{m_0}} (\lambda-1) \\ &= - \frac{3(3n-2)n}{(3n-1)(2n-1)} \operatorname{sech}^2 \frac{x}{2h} \sqrt{\frac{(3n-2)(2n-2)}{(-2n^3+13n^2-15n+5)}} (\lambda-1) \end{aligned}$$

where  $n > 1$ . For  $n > 1$ , both  $m_1$  and  $m_2$  are positive; however, for  $1 < n < n_0 \cong 5.15$ ,  $m_0$  is positive; for  $n = n_0$ ,  $m_0$  is equal to zero; and for  $n > n_0$ ,  $m_0$  is negative. Denote by  $I_1$ ,  $I_2$  the open intervals  $(1, n_0)$  and  $(n_0, \infty)$  respectively. The results are listed in the following table:

	$m_0$	$m_1$	$m_2$	$\lambda-1$	Wave type
$I_1$	+	+	+	+	E
$I_2$	-	+	+	-	D

The sign of  $\lambda-1$  must be such that  $\frac{m_2}{m_0} (\lambda-1)$  is always positive. The last column of the table indicates the wave



type of the solitary wave where we use  $E$  for a wave of elevation and  $D$  for a wave of depression. As seen from the expression for  $\bar{f}$  in (4.9) the wave type is determined by the sign of  $(\lambda-1) \frac{m_2}{m_1}$ . For  $n = 1$  or  $n_0$ , the solitary wave solution does not exist since  $a_1(\sigma) \equiv 0$  for both cases. It also follows from the expression for  $\bar{f}$  that the maximum deviation of the stream lines from the horizontal occurs at the free surface.



## Part II. Compressible Media of Infinite Depth

### with Two Isothermal Layers

#### 1. Introduction

In the study of the atmosphere, if we neglect the effect of the earth rotation and curvature, the atmosphere may be regarded as a medium consisting of a series of isothermal layers extended to infinity over a plane level surface. There exists a linear relation between the pressure and the density in each layer if the equation of state for a perfect gas is used. Each layer is separated from the other by a contact surface across which pressure and velocity are continuous but density and temperature are subject to a jump. Let us suppose that the medium is at rest initially and a solitary wave has been created by some disturbance. We may choose a moving coordinate system with respect to which the flow becomes stationary. Now we have a set of equations as given in Part I govern the flow in each layer and two boundary conditions at the bottom and the free surface. At each interface there are two conditions of continuity to match the solutions for each layer. However, the difficulty lies not in finding the solution but rather in the interpretation of the results. This will be seen later even for the two-layer case we are going to discuss. Nevertheless, the analysis presented here will serve the purpose of illustrating the method of approach.



We shall consider the problem of solitary waves in a compressible medium of two isothermal layers separated by a contact surface. The temperature of the lower layer is always assumed to be less than that of the upper layer. In meteorological terminology this situation may correspond to the so-called "thermal inversion." The problem is formulated in Section 2. In Section 3 the analysis based upon the linearized equations is presented and a solution in closed form is obtained. The linear theory predicts two critical speeds for a given equilibrium state.

In Section 4 we return to the nonlinear theory as we did in Part I. The two critical speeds obtained confirm those due to the linear theory and the coefficients of the solitary wave solution indicate a complicated flow pattern in relation to the parameters defined. The discussion of the results will be given in Section 5. The method employed in this part is quite similar to the one of Part I; however, the analysis given here is self-contained with minimum reference to the previous results.

## 2. Formulation of the Problem

Let us assume that a mass of compressible medium consisting of two layers fills up the whole upper half space. The lower layer, supported by a rigid plane bottom, is at temperature  $T_1$  and of an equilibrium height  $h$ ; the upper layer at temperature



$T_2$  separated from the lower layer by a contact surface is extended to infinity in the state of equilibrium. The pressure  $p$  is assumed to be zero at infinity and there are no geometric constraints. A cross section of the medium at equilibrium is just the upper half plane as shown in the Figure 2. We suppose that a wave of permanent type moving to the left with constant velocity  $c$  has been created in the medium initially at rest. A coordinate system moving with the wave is chosen such that the  $x$ -axis coincides with the bottom and the  $y$ -axis passes through the crest or the trough of the wave and is positive upward (Fig. 2). As observed from the coordinate system, the wave is stationary and the medium in the state of equilibrium at infinity moves to the right with constant velocity  $c$ .

The governing equations for the lower layer are

$$\begin{aligned}
 & \frac{\partial(\tilde{\rho}\tilde{u})}{\partial x} + \frac{\partial(\tilde{\rho}\tilde{v})}{\partial y} = 0, \\
 & \tilde{u} \frac{\partial\tilde{u}}{\partial x} + \tilde{v} \frac{\partial\tilde{u}}{\partial y} = - \frac{1}{\tilde{\rho}} \frac{\partial\tilde{p}}{\partial x}, \\
 & \tilde{u} \frac{\partial\tilde{v}}{\partial x} + \tilde{v} \frac{\partial\tilde{v}}{\partial y} = - g - \frac{1}{\tilde{\rho}} \frac{\partial\tilde{p}}{\partial y}, \\
 & \frac{\tilde{p}}{\tilde{\rho}_0} = \frac{\tilde{p}}{\tilde{\rho}_0},
 \end{aligned}
 \tag{2.1}$$

where  $\tilde{\rho}(x,y)$  is the density,  $\tilde{u}(x,y)$ ,  $\tilde{v}(x,y)$  are the horizontal and vertical velocity components,  $\tilde{p}(x,y)$  is the pressure,



$g$  is the gravitational constant, and  $\tilde{\rho}_0$ ,  $\tilde{p}_0$  are the values of  $\rho$  and  $p$  at  $y = 0$  in the equilibrium state. The same equations also hold for the upper layer.

We first consider the equilibrium state of the medium moving with constant velocity  $\tilde{u}_\infty = c$ , where the subscript  $\infty$  will always denote the quantities in the equilibrium state. It is seen from (2.1) that all the state variables are functions of  $y$  only. Suppose that  $\tilde{\rho}_0$ ,  $\tilde{p}_0$ , the values of  $\tilde{p}$  and  $\tilde{\rho}$  at  $y = 0$ , are given, and at the interface  $y = h$   $\tilde{p} = \tilde{p}_1$ ,  $\tilde{\rho} = \tilde{\rho}_1$  at  $y = h^-$  and  $\tilde{p} = \tilde{p}_2$  at  $y = h^+$ , at infinity  $\tilde{p} = \tilde{p} = 0$ . We obtain from (2.1) that for  $0 \leq y < h$

$$\tilde{\rho}_\infty = \tilde{\rho}_1 \exp\left[-\frac{g\tilde{\rho}_1}{\tilde{p}_1}(y-h)\right]$$

and for  $h < y < \infty$

$$\tilde{\rho}_\infty = \tilde{\rho}_2 \exp\left[-\frac{g\tilde{\rho}_2}{\tilde{p}_1}(y-h)\right].$$

Denote by  $\tilde{\psi}(x, y)$  the stream function such that

$$\tilde{\rho}\tilde{u} = \tilde{\psi}_y, \quad \tilde{\rho}\tilde{v} = -\tilde{\psi}_x,$$

then the mass flux across any vertical plane from  $y = 0$  to  $y = h$  per unit breadth is given by

$$\tilde{\psi}(x, h) - \tilde{\psi}(x, 0) = \int_0^h \tilde{\rho}_\infty \tilde{u}_\infty dy = \tilde{\rho}_1 c H_1$$



where  $H_1 = M_1 [\exp(M_1)^{-1} - 1]h$  ,

$$M_1 = \frac{\tilde{p}_1 / \tilde{\rho}_1}{gh}$$

and the mass flux across any vertical plane from  $y = h$  to  $y = \infty$  per unit breadth is

$$\tilde{\psi}(x, \infty) - \tilde{\psi}(x, h) = \int_h^\infty \tilde{\rho}_1 \tilde{u}_\infty dy = \tilde{\rho}_2 c H_2$$

where  $H_2 = M_2 h$

$$M_2 = \frac{\tilde{p}_1 / \tilde{\rho}_2}{gh} .$$

We may choose

$$\tilde{\psi}(x, 0) = 0 ,$$

and it follows that

$$\tilde{\psi}(x, h) = \tilde{\rho}_1 c H_1 ,$$

$$\tilde{\psi}(x, \infty) = \tilde{\rho}_2 c H_2 .$$

Let the totality of stream lines be given by

$$\tilde{\psi}(x, y) = \gamma$$

where  $0 \leq y < \infty$  ,  $0 \leq \gamma < \tilde{\rho}_2 c H_2$  . It is assumed that for each  $\gamma$  there exists one and only one stream line such that

$$y = \bar{f}(x, \gamma) .$$



If  $x$ ,  $\gamma$  are chosen as independent variables, and  $\bar{f}$ ,  $\bar{p}$ ,  $\bar{\rho}$  and  $\bar{u}$ , as dependent variables where the bar notation indicates a function of  $x$  and  $\gamma$ , as we did in Part I, we obtain

for  $0 \leq \gamma < \tilde{\rho}_1 cH_1$ ,  $-\infty < x < \infty$ ,

$$\bar{u}_x = -\bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma,$$

$$\bar{u} \bar{f}_{xx} + \bar{u}_x \bar{f}_x = -\bar{\rho} g \bar{f}_\gamma - \bar{p}_\gamma,$$

$$\bar{\rho} \bar{u} \bar{f}_\gamma = 1,$$

$$\frac{\bar{p}}{\tilde{\rho}_1} = \frac{\bar{\rho}}{\tilde{\rho}_1};$$

for  $\tilde{\rho}_1 cH_1 < \gamma < \tilde{\rho}_2 cH_2 + \tilde{\rho}_1 cH_1$ ,  $-\infty < x < \infty$ ,

(2.2)

$$\bar{U}_x = -\bar{F}_\gamma \bar{P}_x + \bar{F}_x \bar{P}_\gamma,$$

$$\bar{U} \bar{F}_{xx} + \bar{U}_x \bar{F}_x = -\bar{\Delta} g \bar{F}_\gamma - \bar{P}_\gamma,$$

$$\bar{\Delta} \bar{U} \bar{F}_\gamma = 1,$$

$$\frac{\bar{P}}{\tilde{\rho}_1} = \frac{\bar{\Delta}}{\tilde{\rho}_2};$$

at the bottom,  $\bar{f}(x, 0) = 0$ ,



$$\text{at } \gamma = \gamma_1 = \tilde{\rho}_2 c H_2 + \tilde{\rho}_1 c H_1, \quad \bar{P}(x, \gamma_0) = 0,$$

$$\text{and at the interface, } \gamma = \gamma_0 = \tilde{\rho}_1 c H_1$$

$$\bar{f}(x, \gamma_0) = \bar{F}(x, \gamma_0), \quad \bar{p}(x, \gamma_0) = \bar{P}(x, \gamma_0).$$

If we introduce the following dimensionless variables

$$\xi = \frac{x}{H_1}, \quad \eta = \frac{\gamma}{\tilde{\rho}_0 c H_1}, \quad v = \frac{\bar{v}}{c},$$

$$u = \frac{\bar{u}}{c}, \quad p = \frac{\bar{p}}{\tilde{\rho}_1 c^2}, \quad \rho = \frac{\bar{\rho}}{\tilde{\rho}_1},$$

$$f = \frac{\bar{f}}{H_1}, \quad V = \frac{\bar{V}}{c}, \quad U = \frac{\bar{U}}{c},$$

$$P = \frac{\bar{P}}{\tilde{\rho}_1 c^2}, \quad \Delta = \frac{\bar{\Delta}}{\tilde{\rho}_2}, \quad F = \frac{\bar{F}}{H_1},$$

$$M = [\exp(M_1)^{-1} - 1]^{-1}, \quad r = 1 + M,$$

$$\beta = \frac{\tilde{\rho}_2}{\tilde{\rho}_1} = \frac{T_1}{T_2} < 1, \quad \alpha = \frac{1}{\beta}, \quad \lambda = \frac{g H_1}{c^2},$$

the equations (2.2) become

$$\text{for } 0 \leq \eta < 1, \quad -\infty < \xi < \infty,$$

$$u_\xi = -p_\xi f_\eta + f_\xi p_\eta,$$



$$(2.3) \quad u f_{\xi\xi} + u_{\xi} f_{\xi} = - \lambda \rho f_{\eta} - p_{\eta} ,$$

$$\rho u f_{\eta} = 1 ,$$

$$p = M \lambda \rho ;$$

$$\text{for } 1 < \eta < r , \quad -\infty < \xi < \infty ,$$

$$\overline{U}_{\xi} = - P_{\xi} F_{\eta} + F_{\xi} P_{\eta} ,$$

$$U F_{\xi\xi} + U_{\xi} F_{\xi} = - \beta \lambda \Delta F_{\eta} - P_{\eta} ,$$

$$\Delta U F_{\eta} = \alpha ,$$

$$P = M \lambda \Delta ,$$

with the boundary conditions,

$$f(\xi, 0) = 0 ,$$

$$P(\xi, r) = 0 ,$$

$$f(\xi, 1) = F(\xi, 1) ,$$

$$p(\xi, 1) = P(\xi, 1) .$$

Here we note that the dependent variables denoted by lower case letters always correspond to the lower layer and those denoted by capital letters always correspond to the upper layer. It is also understood that the values of the variables at the branch line  $\eta = r$  are defined as the limits of these variables as  $\eta \uparrow r$ .



### 3. Linear Theory. Critical Speeds

For the equilibrium state of a steady flow with  $u = U \equiv 1$  we find from (2.3) that

$$p_0 = \lambda(r-\eta) , \quad P_0 = \lambda(r-\eta) ,$$

$$\rho_0 = (M)^{-1} (r-\eta) , \quad \Delta_0 = (M)^{-1} (r-\eta) ,$$

$$f_0 = -M \log \frac{r-\eta}{r} , \quad F_0 = -M\alpha \log \frac{r-\eta}{r-1} - M \log \frac{r-1}{r} .$$

Let us consider a small disturbance superposed on the equilibrium state and we may write

$$p = \lambda(r-\eta) + p^* , \quad P = \lambda(r-\eta) + P^* ,$$

$$\rho = (M)^{-1} (r-\eta) + \rho^* , \quad \Delta = (M)^{-1} (r-\eta) + \Delta^* ,$$

$$f = -M \log \frac{r-\eta}{r} + f^* , \quad F = -M\alpha \log \frac{r-\eta}{r-1} - M \log \frac{r-1}{r} + F^* .$$

We substitute the above quantities in (2.3) and assume that all the terms involving the second order products of the starred quantities can be neglected. The linearized equations are:

for  $0 < \eta < 1$  ,  $-\infty < \xi < \infty$  ,

$$u_\xi^* = -p_\xi^* f_{0\eta} + f_\xi^* p_{0\eta} ,$$



$$f_{\xi\xi}^* = -\lambda(\rho_0 f_{\eta}^* + \rho^* f_{0\eta}) - p_{\eta}^* ,$$

$$u^* + \rho^* f_{0\eta} + \rho_0 f_{\eta}^* = 0 ,$$

$$p^* = M\lambda\rho^* :$$

for  $1 < \eta < r$  ,  $-\infty < \xi < \infty$  ,

$$(2.4) \quad U_{\xi}^* = -P_{\xi}^* F_{0\eta} + F_{\xi}^* P_{0\eta} ,$$

$$F_{\xi\xi}^* = -\beta\lambda(\Delta_0 F_{\eta}^* + \Delta^* F_{0\eta}) - P_{\eta}^*$$

$$\alpha U^* + \Delta^* F_{0\eta} + \Delta_0 F_{\eta}^* = 0 ,$$

$$P^* = M\lambda\Delta^* .$$

together with the boundary conditions

$$f^*(\xi, 0) = 0 ,$$

$$P^*(\xi, r) = 0 ,$$

$$f^*(\xi, 1) = F^*(\xi, 1) ,$$

$$p^*(\xi, 1) = P^*(\xi, 1) .$$

From (2.4) we find that for  $0 < \eta < 1$

$$f_{\xi\xi\xi}^* = -M^{-1}\lambda(M\lambda - 1)^{-1}(r-\eta)^2 f_{\xi\eta\eta}^* + 2M^{-1}\lambda(M\lambda - 1)^{-1}(r-\eta) f_{\xi\eta}^* ,$$

$$\rho_{\xi}^* = [(M\lambda - 1) f_{0\eta}]^{-1}(\rho_0 f_{\xi\eta}^* + f_{\xi}^* p_{0\eta}) ;$$

and for  $1 < \eta < r$  .



(2.5)

$$F_{\xi\xi\xi}^* = - \frac{\lambda}{M\alpha(\alpha M\lambda - 1)} (r-\eta)^2 F_{\xi\eta\eta}^* + \frac{2\lambda}{M\alpha(\alpha M\lambda - 1)} (r-\eta) F_{\xi\eta}^* ,$$

$$\Delta_{\xi}^* = [(\alpha M\lambda - 1) F_{0\eta}]^{-1} (\Delta_0 F_{\xi\eta}^* + \alpha F_{\xi}^* P_{0\eta}) ,$$

subject to

$$f_{\xi}^*(\xi, 0) = 0 ,$$

$$\Delta_{\xi}^*(\xi, r) = 0 ,$$

$$f_{\xi}^*(\xi, 1) = F_{\xi}^*(\xi, 1) ,$$

$$\rho_{\xi}^*(\xi, 1) = \Delta_{\xi}^*(\xi, 1) .$$

The solutions of (2.5) which satisfy the boundary conditions at  $\eta = 0$  and  $\eta = r$  are

$$f_{\xi}^* = a_1 \cos(v\xi + b) [(r-\eta)^{s_1} - r^{2\omega}(r-\eta)^{s_2}] ,$$

$$\rho_{\xi}^* = a_1 \cos(v\xi + b) \frac{r-\eta}{M(M\lambda - 1)} [-(M^{-1}s_1 + \lambda)(r-\eta)^{s_1} +$$

$$r^{2\omega}(M^{-1}s_2 + \lambda)(r-\eta)^{s_1}] ,$$

$$F_{\xi}^* = a_2 \cos(v\xi + b) (r-\eta)^{s_1}$$

$$\Delta_{\xi}^* = - a_2 \cos(v\xi + b) \frac{M^{-1}s_1 + \alpha\lambda}{M\alpha(\alpha M\lambda - 1)} (r-\eta)^{s_1+1} ,$$

where  $\omega = \frac{1}{2} [1 + \frac{4(M\lambda - 1)M}{\lambda} v^2]^{1/2}$

$$s_1 = -\frac{1}{2} + \omega , \quad s_2 = -\frac{1}{2} - \omega ,$$



$$S_1 = \frac{1}{2} \left[ -1 + \left( 1 + \frac{4(\alpha M \lambda - 1) M \alpha}{\lambda} v^2 \right)^{1/2} \right],$$

and  $a_1$ ,  $a_2$ , and  $b$  are arbitrary constants. Application of the boundary conditions at  $\xi = 1$  yields

$$\begin{aligned} & - (M^{-1} s_1 + \lambda)(r-1)^{s_1} + r^{2\omega} (M^{-1} s_2 + \lambda)(r-1)^{s_2} \\ & = - [(r-1)^{s_1} - r^{2\omega} (r-1)^{s_2}] (M^{-1} s_1 + \alpha \lambda) \frac{(M \lambda - 1)}{\alpha(\alpha M \lambda - 1)}. \end{aligned}$$

Now we define the critical speed  $\ell$  as the limiting value of  $\lambda$  when  $v \rightarrow 0$ . We obtain from the above equation

$$(1 - M\ell) = (\alpha M \ell - 1) \left( \frac{r}{M\ell} - 1 \right),$$

and

$$\ell = \frac{\alpha r \pm [(\alpha r)^2 - 4r(\alpha - 1)]^{1/2}}{2(\alpha - 1)M}.$$

We see that for given values of  $\alpha$  and  $M$  we always have two critical speeds. It will be shown in Section 5 that  $0 < \ell_- M < 1$ ,  $1 < \ell_+ M < \infty$  if  $1 < \alpha < \infty$ ,  $1 < r < \infty$ , where  $\ell_-$  corresponds to the "-" sign and  $\ell_+$  the "+" sign in the expression for  $\ell$ . Furthermore, if we rewrite the quadratic equation for  $\ell$  as

$$\alpha \ell M = \frac{(\alpha - 1)(\ell M)^2}{r},$$

it is seen that  $\alpha \ell M - 1$  is always positive for  $1 < \alpha < \infty$  and  $1 < r < \infty$ . Therefore for small values of  $v$ ,  $\lambda M - 1 \neq 0$  and  $\alpha \lambda M - 1 > 0$ .



Let us now consider the case  $\alpha = 1$  .  $1 < r < \infty$  . First suppose that  $M\lambda \neq 1$  . We obtain

$$f_{\xi}^* = F_{\xi}^* = a_1 \cos(k\xi + b) [(r-\eta)^{s_1} - r^{2\omega}(r-\eta)^{s_2}]$$

which is unbounded at  $\eta = r$  as seen from the expressions for  $s_1$  and  $s_2$  . Therefore, the linear theory fails, otherwise we must set  $a_1 = 0$  . If  $M\lambda = 1$  , from (2.5), we obtain

$$(r-\eta)f_{\xi\eta}^* - f_{\xi}^* = 0 ;$$

since  $f_{\xi}^*(\xi, 0) = 0$  ,  $f_{\xi}^*$  must be identically equal to zero. The above results confirm what we already established in Part I. For an infinite isothermal layer, either the linear theory fails or the solution is a trivial one.

Finally we shall justify the consistency of our linearizing procedure. Since  $f_{\xi}^*$  ,  $\rho_{\xi}^*$  are bounded for  $0 \leq \eta \leq 1$  , it will suffice to examine whether  $F_{\xi}^*$  ,  $\Delta_{\xi}^*$  are bounded as  $\eta \rightarrow r^-$  . The terms involving the second order products of the starred quantities, which we have deleted from the equations for the upper layer are:

$$F_{\xi}^* P_{\eta}^* , F_{\eta}^* P_{\xi}^* , U^* F_{\xi\xi}^* , U_{\xi}^* F_{\xi}^* , \Delta^* F_{\eta}^* , F_{0\eta} U^* \Delta^* ,$$

$$\Delta_0 U^* F_{\eta}^* , \Delta^* F_{\eta}^* , \Delta^* U^* F_{\eta}^* .$$

We see from the solutions for  $F_{\xi}^*$  ,  $\Delta_{\xi}^*$  that those terms are



of the order  $(r-\eta)^{2S_1}$  or  $(r-\eta)^{3S_1}$  and tend to zero as  $\eta \rightarrow r^-$  if  $\alpha M\lambda - 1 > 0$ . However, for small values of  $v$ ,  $\alpha M\lambda - 1 > 0$  for  $1 < \alpha < \infty$ ,  $1 < r < \infty$ , and the linearizing procedure is then justified.

#### 4. Nonlinear Theory. Solitary Wave Solution

We assume that a solitary wave moves with a speed such that  $\lambda = \frac{gh}{c^2}$  is near some positive value  $\ell$  which is to be determined later. We write (2.3) in the form, for  $0 \leq \eta < 1$ ,  $-\infty < \xi < \infty$ ,

$$u_\xi = -p_\xi f_\eta + f_\xi p_\eta, \quad (4.1)$$

$$u f_{\xi\xi} + u_\xi f_\xi = (\ell - \lambda)\rho f_\eta - \ell\rho f_\eta - p_\eta.$$

$$\rho u f_\eta = 1,$$

$$p = -M(\ell - \lambda)\rho + M\ell\rho.$$

Similar equations will hold for the upper layer. Now let  $\varepsilon = \ell - \lambda$  and introduce a new independent variable  $\sigma = \xi\sqrt{\varepsilon}$ . Then (4.1) becomes, for  $0 \leq \eta < 1$ ,  $-\infty < \sigma < \infty$ ,

$$u_\sigma = -p_\sigma f_\eta + f_\sigma p_\eta,$$



$$\varepsilon(u f_{\sigma\sigma} + u_{\sigma} f_{\sigma}) = \varepsilon \rho f_{\eta} - \ell \rho f_{\eta} - p_{\eta} ,$$

$$\rho u f_{\eta} = 1 ,$$

$$p = - M \varepsilon \rho + M \ell \rho ;$$

for  $1 < \eta < r$  ,  $-\infty < \sigma < \infty$

$$U_{\sigma} = - P_{\sigma} F_{\eta} + F_{\sigma} P_{\eta}$$

$$(4.2) \quad \varepsilon(U F_{\sigma\sigma} + U_{\sigma} F_{\sigma}) = \varepsilon \beta \Delta F_{\eta} - \beta \ell \Delta F_{\eta} - P_{\eta} .$$

$$\Delta U F_{\eta} = \alpha .$$

$$P = - M \varepsilon \Delta + M \ell \Delta ,$$

with the boundary conditions,

$$f(\sigma, 0) = 0 ,$$

$$P(\sigma, r) = 0 .$$

$$f(\sigma, 1) = F(\sigma, 1) ,$$

$$p(\sigma, 1) = P(\sigma, 1) .$$

It is assumed that all the quantities in the new independent variables can be expanded in integral powers of  $\varepsilon$  , i.e.

$$(4.3) \quad \phi(\sigma, \eta, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \phi_k(\sigma, \eta) ,$$

where  $\phi$  stands for  $p$  ,  $\rho$  ,  $u$  ,  $f$  ,  $P$  ,  $\Delta$  ,  $U$  , and  $F$  .

Substitution of (4.3) in (4.2) will give a sequence of equations



and boundary conditions which these equations must satisfy by equating the coefficients of equal powers of  $\epsilon$ . The values of  $\ell$  and also the solitary wave solution will be determined by solving the equations for the successive approximations.

The equations for the zero-th order approximation are, for  $0 < \eta < 1$ ,  $-\infty < \sigma < \infty$ ,

$$u_{0\sigma} = -p_{0\sigma} f_{0\sigma} + f_{0\sigma} p_{0\eta},$$

$$0 = -\ell \rho_0 f_{0\eta} - p_{0\eta},$$

$$\rho_0 u_0 f_{0\eta} = 1,$$

$$p_0 = M\ell\rho_0;$$

(4.4) for  $1 < \eta < r$ ,  $-\infty < \sigma < \infty$ .

$$U_{0\sigma} = -P_{0\sigma} F_{0\eta} + F_{0\sigma} P_{0\eta}$$

$$0 = -\beta\ell\Delta_0 F_{0\eta} - P_{0\eta}.$$

$$\Delta_0 U_0 F_{0\eta} = \alpha,$$

$$P_0 = M\ell\Delta_0.$$

with the boundary conditions



$$f_0(\sigma, 0) = 0 ,$$

$$P_0(\sigma, r) = 0 ,$$

$$f_0(\sigma, 1) = F_0(\sigma, 1) :$$

$$p_0(\sigma, 1) = P_0(\sigma, 1) .$$

We may assume that  $u_0 = U_0 \equiv 1$  which expresses a parallel flow in the equilibrium state. The solution for the zero-th approximation is

$$\begin{aligned} p_0 &= \ell(r-\eta) , & P_0 &= \ell(r-\eta) , \\ (4.5) \quad \rho_0 &= (M)^{-1}(r-\eta) , & \Delta_0 &= (M)^{-1}(r-\eta) , \\ f_0 &= -M \log \frac{r-\eta}{r} , & F_0 &= -M\alpha \log \frac{r-\eta}{r-1} - M \log \frac{r-1}{r} . \end{aligned}$$

The equations for the first order approximation are for  $0 < \eta < 1$  ,  $-\infty < \sigma < \infty$  ,

$$u_{1\sigma} = -p_{1\sigma} f_{0\eta} + f_{1\sigma} p_{0\eta} ,$$

$$0 = \rho_0 f_{0\eta} - \ell(\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - p_{1\eta} ,$$

$$u_1 = -\rho_0 f_{1\eta} - \rho_1 f_{0\eta} ,$$

$$p_1 = M\ell\rho_1 - Mp_0 ;$$

(4.6) for  $1 < \eta < r$  ,  $-\infty < \sigma < \infty$  ,

$$U_{1\sigma} = -P_{1\sigma} F_{0\eta} + F_{1\sigma} P_{0\eta} ,$$

$$0 = \beta\Delta_0 F_{0\eta} - \beta\ell(\Delta_0 F_{1\eta} + \Delta_1 F_{0\eta}) - P_{1\eta} ,$$



$$\alpha U_1 = - \Delta_0 F_{1\eta} - \Delta_1 F_{0\eta} ,$$

$$P_1 = M\ell\Delta_1 - M\rho_0 ,$$

subject to the boundary conditions

$$f_1(\sigma, 0) = 0 , \quad P_1(\sigma, r) = 0 ,$$

$$f_1(\sigma, 1) = F_1(\sigma, 1) , \quad p(\sigma, 1) = P(\sigma, 1)$$

From (4.6) we have by elimination of  $u_1$  ,  $p_1$  , and  $f_1$  ,

$$\rho_{1\sigma\eta\eta} = 0 ,$$

and then  $f_{1\sigma}$  is obtained by

$$f_{1\sigma} = (p_{0\eta})^{-1} (M\rho_{1\eta\sigma} + M\ell\rho_{1\sigma}f_{0\eta}) .$$

The solutions for  $\rho_{1\sigma}$  and  $f_{1\sigma}$  which satisfy the condition at  $\eta = 0$  are

$$\rho_{1\sigma} = a_1'(\sigma) \left( \frac{r}{M\ell} - \eta \right) ,$$

$$f_{1\sigma} = - \ell^{-1} M a_1'(\sigma) \frac{\eta(1 - M\ell)}{r - \eta} ,$$

where  $a_1'(\sigma)$  is an arbitrary function of  $\sigma$  . Since it is assumed that the flow reaches the state of equilibrium at  $\sigma = -\infty$  i.e.  $a_1(\sigma) \rightarrow 0$  as  $\sigma \rightarrow -\infty$  , we have

$$\rho_1 = a_1(\sigma) \left( \frac{r}{M\ell} - \eta \right) ,$$



$$r_1 = - \ell^{-1} Ma_1(\sigma) \frac{\eta(1 - M\ell)}{r-\eta} ,$$

where we assume that  $a_1(\sigma)$  is not identically equal to zero. Similarly the solutions for  $\Delta_{1\sigma}$  and  $F_{1\sigma}$  which satisfy the condition at  $\eta = r$  are

$$\Delta_{1\sigma} = A_1'(\sigma)(r-\eta) ,$$

$$F_{1\sigma} = - \ell^{-1} MA_1'(\sigma)(-1 + M\ell\alpha) ,$$

where  $A_1'(\sigma)$  is an arbitrary function of  $\sigma$ . By the equilibrium condition at  $x = -\infty$ , we have

$$\Delta_1 = A_1(\sigma)(r-\eta) ,$$

$$F_1 = - \ell^{-1} MA_1(\sigma)(-1 + M\ell\alpha) ,$$

where we assume that  $A_1(\sigma) \neq 0$ . Making use of the conditions at the interface  $\eta = 1$ , we obtain

$$a_1(\sigma)\left(\frac{r}{M\ell} - 1\right) = A_1(\sigma)(r-1) ,$$

(4 7)

$$\ell^{-1} Ma_1(\sigma) \frac{1 - M\ell}{r-1} = \ell^{-1} MA_1(\sigma)(\alpha\ell M - 1) .$$

Since  $A_1(\sigma)$ ,  $a_1(\sigma) \neq 0$ , it follows that

$$(1 - M\ell) = \left(\frac{r}{\ell M} - 1\right)(\alpha\ell M - 1) ,$$

and

$$(4\ 8) \quad \ell = \frac{\alpha r \pm \sqrt{(\alpha r)^2 - 4r(\alpha-1)}}{2(\alpha-1)M} .$$



Thus, we have two values of  $\ell$  which confirm the two critical speeds obtained from the linear theory. The results for the first-order approximation are summarized as below:

(4.9)

$$\rho_1 = a_1(\sigma)\left(\frac{r}{M\ell} - \eta\right), \quad \Delta_1 = A_1(\sigma)(r-\eta),$$

$$p_1 = M\ell a_1(\sigma)\left(\frac{r}{M\ell} - \eta\right) - (r-\eta), \quad P_1 = (M\ell A_1(\sigma) - 1)(r-\eta),$$

$$f_1 = -\ell^{-1}Ma_1(\sigma) \frac{(1 - M\ell)\eta}{r-\eta}, \quad F_1 = -\ell^{-1}MA_1(\sigma)(\alpha\ell M - 1),$$

$$u_1 = -Ma_1(\sigma), \quad U_1 = -MA_1(\sigma).$$

In order to determine  $a_1(\sigma)$  and  $A_1(\sigma)$ , we must proceed to the equations for the second order approximation, which are found as follows:

for  $0 < \eta < 1$ ,  $-\infty < \sigma < \infty$ ,

$$u_{2\sigma} = -(p_{2\sigma}f_{0\eta} + p_{1\sigma}f_{1\eta}) + (f_{2\sigma}p_{0\eta} + f_{1\sigma}p_{1\eta}),$$

$$f_{1\sigma\sigma} = (\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - \ell(\rho_0 f_{2\eta} + \rho_1 f_{1\eta} + \rho_2 f_{0\eta}) - p_{2\eta},$$

$$\rho_0 u_{2\sigma} f_{0\eta} + \rho_0 u_{0\sigma} f_{2\eta} + \rho_2 u_{2\sigma} f_{0\eta} + \rho_0 u_{1\sigma} f_{1\eta} + \rho_1 u_{0\sigma} f_{1\eta} + \rho_1 u_{1\sigma} f_{0\eta} = 0,$$

$$p_2 = M\ell\rho_2 - Mp_1;$$

(4.10)

for  $1 < \eta < r$ ,  $-\infty < \sigma < \infty$ ,

$$U_{2\sigma} = -P_{2\sigma}F_{0\eta} + F_{2\sigma}P_{0\eta} + F_{1\sigma}P_{1\eta},$$



$$F_{1\sigma\sigma} = \beta(\Delta_0 F_{1\eta} + \Delta_1 F_{0\eta}) - \beta\ell(\Delta_0 F_{2\eta} + \Delta_2 F_{0\eta} + \Delta_1 F_{1\eta}) - P_{2\eta} ,$$

$$\Delta_0 U_2 F_{0\eta} + \Delta_0 U_0 F_{2\eta} + \Delta_2 U_0 F_{2\eta} + \Delta_1 U_0 F_{1\eta} + \Delta_0 U_1 F_{1\eta} + \Delta_1 U_1 F_{0\eta} = 0 ,$$

$$P_2 = M\ell\Delta_2 - M\Delta_1 ,$$

together with the boundary conditions

$$f_2(\sigma, 0) = 0 ,$$

$$P_2(\sigma, r) = 0 ,$$

$$f_2(\sigma, 1) = F_2(\sigma, 1) ,$$

$$p_2(\sigma, 1) = P_2(\sigma, 1) .$$

For the lower layer, we find that

$$\rho_{2\sigma\eta\eta} = \frac{1}{M} g_4(\sigma, \eta) ,$$

(4.11)

$$f_{2\sigma} = (p_{0\eta})^{-1}(Mp_{2\sigma\eta} + \ell^{-1}g_{2\sigma}(\sigma, \eta) + M\ell\rho_{2\sigma}f_{0\eta} - g_1(\sigma, \eta))$$

where

$$g_4(\sigma, \eta) = - \ell^{-1}g_{2\sigma\eta}(\sigma, \eta) - \rho_0^{-1}p_{0\eta}\ell^{-1}g_{2\sigma}(\sigma, \eta) + \rho_0^{-1}p_{0\eta} \times$$

$$\times g_{3\sigma}(\sigma, \eta) + g_{1\eta}(\sigma, \eta) ,$$

$$g_1(\sigma, \eta) = Mp_{1\sigma}f_{0\eta} - p_{1\sigma}f_{1\eta} + f_{1\sigma}p_{1\eta} ,$$

(4.12)

$$g_2(\sigma, \eta) = u_1 + \ell u_1^2 - Mp_{1\eta} + f_{1\sigma\sigma} ,$$

$$g_3(\sigma, \eta) = - \rho_1 f_{1\eta} + u_1^2 .$$



Let

$$g_4^*(\sigma, \eta) = \int_0^\eta g_4(\sigma, \eta') d\eta' ,$$

$$g_4^{**}(\sigma, \eta) = \int_0^\eta \int_0^{\eta'} g_4(\sigma, \eta') d\eta' d\eta'' .$$

We have from (4.11)

$$\rho_{2\sigma} = M^{-1} g_4^{**}(\sigma, \eta) - a_2'(\sigma) \eta + b_2'(\sigma) ,$$

$$\begin{aligned} f_{2\sigma} &= (p_{0\eta})^{-1} [g_4^*(\sigma, \eta) - M a_2'(\sigma) + \ell f_{0\eta} g_4^{**}(\sigma, \eta) \\ &\quad - M \ell f_{0\eta} a_2'(\sigma) \eta + M \ell f_{0\eta} b_2'(\sigma) + \ell^{-1} g_{2\sigma}(\sigma, \eta) - g_1(\sigma, \eta)] , \end{aligned}$$

where  $a_2'(\sigma)$  and  $b_2'(\sigma)$  are two arbitrary functions of  $\sigma$ .

Since  $f_{2\sigma}(\sigma, 0) = 0$ , by (4.5) it is obtained that

$$\begin{aligned} b_2'(\sigma) &= \frac{r}{M\ell} a_2'(\sigma) - \frac{r}{M^2\ell} g_4^*(\sigma, 0) - \frac{1}{M} g_4^{**}(\sigma, 0) \\ &\quad - \frac{r}{(\ell M)^2} g_{2\sigma}(\sigma, 0) + \frac{r}{M^2\ell} g_1(\sigma, 0) . \end{aligned}$$

Hence,

$$\begin{aligned} \rho_{2\sigma} &= \left(\frac{r}{M\ell} - \eta\right) a_2'(\sigma) + M^{-1} g_4^{**}(\sigma, \eta) - \frac{r}{M^2\ell} g_4^*(\sigma, 0) , \\ &\quad - M^{-1} g_4^{**}(\sigma, 0) - \frac{r}{(\ell M)^2} g_{2\sigma}(\sigma, 0) + \frac{r}{M^2\ell} g_1(\sigma, 0) , \\ (4.13) \quad f_{2\sigma} &= - \ell^{-1} [- M a_2'(\sigma) - M \ell f_{0\eta} \eta a_2'(\sigma) + r f_{0\eta} a_2'(\sigma) + g_4^*(\sigma, \eta) \\ &\quad + \ell f_{0\eta} g_4^{**}(\sigma, \eta) + \ell^{-1} g_{2\sigma}(\sigma, \eta) - g_1(\sigma, \eta) - \frac{r}{M} f_{0\eta} g_4^*(\sigma, 0) \\ &\quad - \ell f_{0\eta} g_4^{**}(\sigma, 0) - \frac{r}{\ell M} f_{0\eta} g_{2\sigma}(\sigma, 0) + \frac{r}{M} f_{0\eta} g_1(\sigma, 0)] . \end{aligned}$$



For the upper layer we proceed in the same way, and obtain that

$$\Delta_{2\sigma\eta\eta} = \frac{1}{M} G_4(\sigma, \eta) , \quad (4.14)$$

$$F_{2\sigma} = P_{0\eta}^{-1} [M\Delta_{2\sigma\eta} + \ell^{-1}G_{2\sigma}(\sigma, \eta) + M\ell F_{0\eta}\Delta_{2\sigma} - G_1(\sigma, \eta)] ,$$

where

$$\begin{aligned} G_4(\sigma, \eta) = & -\ell^{-1}G_{2\sigma\eta}(\sigma, \eta) - \Delta_0^{-1}P_{0\eta}\alpha\ell^{-1}G_{2\sigma}(\sigma, \eta) \\ & + \Delta_0^{-1}P_{0\eta}G_{3\sigma}(\sigma, \eta) + G_{1\eta}(\sigma, \eta) , \end{aligned} \quad (4.15)$$

$$G_1(\sigma, \eta) = M\Delta_{1\sigma}F_{0\eta} + F_{1\sigma}P_{1\eta} ,$$

$$G_2(\sigma, \eta) = U_1 + \ell U_1^2 - M\Delta_{1\eta} + F_{1\sigma\sigma} ,$$

$$G_3(\sigma, \eta) = -\Delta_1 F_{1\eta} + \alpha U_1^2 .$$

Let

$$G_4^*(\sigma, \eta) = \int_0^\eta G_4(\sigma, \eta') d\eta' ,$$

$$G_4^{**}(\sigma, \eta) = \int_0^\eta \int_0^{\eta'} G_4(\sigma, \eta') d\eta' d\eta'' .$$

It is obtained from (4.14) that

$$\Delta_{2\sigma} = M^{-1}G_4^{**}(\sigma, \eta) - A_2'(\sigma)\eta + B_2'(\sigma) .$$

Since  $\Delta_{2\sigma}(\sigma, r) = 0$  , it follows that



$$\Delta_{2\sigma} = A'_2(\sigma)(r-\eta) + M^{-1}G^{**}(\sigma, \eta) - M^{-1}G^{**}(\sigma, r) ,$$

(4.16)

$$F_{2\sigma} = F_{0\eta}^{-1} [G^*(\sigma, \eta) - MA'_2(\sigma) + M\ell F_{0\eta} A'_1(\sigma)(r-\eta) \\ + \ell F_{0\eta} G_4^{**}(\sigma, \eta) - \ell F_{0\eta} G_4^{**}(\sigma, r) + \ell^{-1} G_{2\sigma}(\sigma, \eta) - G_1(\sigma, \eta)] .$$

The matching conditions at the interface  $\eta = 1$

$$f_{2\sigma}(\sigma, 1) = F_{2\sigma}(\sigma, 1) , \quad \rho_{2\sigma}(\sigma, 1) = \Delta_{2\sigma}(\sigma, 1)$$

then give two relations between  $A'_2(\sigma)$  and  $a'_2(\sigma)$ , i.e.

$$(\frac{r}{M\ell} - 1)a'_2(\sigma) - (r-1)A'_2(\sigma) = \mathcal{C}_1(\sigma) ,$$

(4.17)

$$M(\frac{1}{r-1} - \frac{M\ell}{r-1})a'_2(\sigma) - M(-1 + M\ell\alpha)A'_2(\sigma) = \mathcal{C}_2(\sigma) ,$$

where

$$\mathcal{C}_1(\sigma) = M^{-1}G_4^{**}(\sigma, 1) - M^{-1}G_4^{**}(\sigma, r) - M^{-1}g_4^{**}(\sigma, 1) \\ + M^{-1}g_4^{**}(\sigma, 0) + \frac{r}{M^2\ell} g_4^*(\sigma, 0) + \frac{r}{(\ell M)^2} g_{2\sigma}(\sigma, 0) \\ - \frac{r}{M^2\ell} g_1(\sigma, 0) ,$$

$$\mathcal{C}_2(\sigma) = - g_4^*(\sigma, 1) + \frac{r}{r-1} g_4^*(\sigma, 0) - \frac{M\ell}{r-1} g_4^{**}(\sigma, 1) \\ + \frac{M\ell}{r-1} g_4^{**}(\sigma, 0) - \ell^{-1} g_{2\sigma}(\sigma, 1) + \frac{r}{\ell(r-1)} g_{2\sigma}(\sigma, 0) \\ + g_1(\sigma, 1) + \frac{r}{r-1} g_1(\sigma, 0) + G_4^*(\sigma, 1) + \frac{M\ell\alpha}{r-1} G_4^{**}(\sigma, 1) \\ - \frac{M\ell\alpha}{r-1} G_4(\sigma, r) + \ell^{-1} G_{2\sigma}(\sigma, 1) - G_1(\sigma, 1) .$$



From the results of (4.7) , the following relation

$$\frac{\frac{r}{M\ell} - 1}{M \frac{1 - M\ell}{r-1}} = \frac{r-1}{M(\alpha\ell M - 1)} = \frac{C_1(\sigma)}{C_2(\sigma)}$$

must hold in order to ensure a consistent system of equations for  $a_2'(\sigma)$  and  $A_2'(\sigma)$  . Let  $\ell$  assume the values given by (4.8) . Then we have

$$(4.18) \quad M(\alpha\ell M - 1)C_1(\sigma) = (r-1)C_2(\sigma) .$$

By some complicated but straightforward computations and rearrangements of the terms, we finally obtain the equation

$$m_0 a_1'''(\sigma) + m_1 a_1'(\sigma) a_1(\sigma) + m_2 a_1'(\sigma) = 0 ,$$

where

$$m_0 = \frac{M^3(1 - \ell M)}{(\ell M)^3} [2(1-r)r(1 - \alpha\ell M)\log \frac{r}{r-1} \\ + 1 + r + \ell M(1 - 2\alpha r)] ,$$

(4.19)

$$m_1 = \frac{r-1}{(\ell M)^2} \left\{ 4r(r-1)\log \frac{r}{r-1} [(\alpha-1)(\ell M)^4 - \alpha(\ell M)^3 + (\ell M)^2] \right. \\ - 2r(\alpha-1)(\ell M)^4 + (-2r^2\alpha + 6r\alpha + \alpha - r + 1)(\ell M)^3 \\ + (\alpha r^2 - 5\alpha r + 4r^2 - 10r + 4)(\ell M)^2 + (4\alpha r^2 - 3\alpha r - 3r^2 \\ \left. + 12r)\ell M - (6r^2 - 3r) \right\} ,$$

$$m_2 = \frac{M^2 r}{\ell M} \left[ \frac{2}{\ell M} - (2+\alpha) \right] .$$



Let us assume that  $m_0$ ,  $m_1$ , and  $m_2$  are not equal to zero for given values of  $r$ ,  $\alpha$ , and since we are only concerned with the solitary wave solution, the following conditions

$$a_1'(-\infty) = a_1''(-\infty) = 0, \quad a_1'(0) = 0$$

are imposed. The solution for  $a_1(\sigma)$  subject to the above conditions is

$$(4.20) \quad a_1(\sigma) = -\frac{3m_2}{m_1} \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{-\frac{m_2}{m_0}}.$$

Then by (4.7) we have

$$(4.21) \quad A_1(\sigma) = -\frac{\frac{r}{M\ell} - 1}{r-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{-\frac{m_2}{m_0}}.$$

Now suppose that the successive approximations so far obtained do furnish a sufficiently accurate representation of a solitary wave in a compressible medium of two isothermal layers, we obtain, in terms of the independent variables  $x$  and  $\gamma$ ,

for  $0 \leq \eta \leq 1$ ,  $-\infty < \kappa < \infty$ ,

$$\bar{p} \approx \tilde{p}_1 g H_1 (r-\eta) - \tilde{p}_1 c^2 (\ell-\lambda) M \ell \left( \frac{r}{M\ell} - \eta \right) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}},$$

$$\bar{\rho} \approx M^{-1} \tilde{\rho}_1 (r-\eta) - \tilde{\rho}_1 (\ell-\lambda) \left( \frac{r}{M\ell} - \eta \right) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}},$$

$$\bar{u} \approx c + M c (\ell-\lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}},$$

$$\bar{f} \approx -M H_1 \log \frac{r-\eta}{r} + \ell^{-1} M H_1 \frac{\eta(1-M\ell)}{r-\eta} (\ell-\lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}},$$



$$\bar{v} \cong c\ell^{-1}M \frac{(1 - M\ell)3m_2}{m_1} \frac{\eta}{r-\eta} (\ell-\lambda) \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} x \\ \times \tanh \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} ;$$

for  $1 \leq \eta \leq r$  ,  $-\infty < \sigma < \infty$  ,

(4.22)

$$\bar{P} \cong \tilde{\rho}_1 g H_1 (r-\eta) - \tilde{\rho}_1 c^2 (\ell-\lambda) (r-\eta) M\ell \frac{\frac{r}{M\ell} - 1}{r-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} ,$$

$$\bar{\Delta} \cong M^{-1} \tilde{\rho}_2 (r-\eta) - \tilde{\rho}_2 (\ell-\lambda) (r-\eta) \frac{\frac{r}{M\ell} - 1}{r-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} ,$$

$$\bar{U} \cong C + MC(\ell-\lambda) \frac{\frac{r}{M\ell} - 1}{r-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} ,$$

$$\bar{F} \cong -MH_1 \alpha \log \frac{r-\eta}{r-1} - MH_1 \log \frac{r-1}{r} + \ell^{-1} MH_1 \frac{(1 - M\ell)}{(r-1)} (\ell-\lambda) x \\ \times \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} ,$$

$$\bar{V} \cong c\ell^{-1}M \frac{(1 - M\ell)}{r-1} \frac{3m_2}{m_1} (\ell-\lambda) \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} \operatorname{sech}^2 \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} x \\ \times \tanh \frac{x}{2H_1} \sqrt{\frac{m_2(\lambda-\ell)}{m_0}} .$$

Where

$$\eta = \frac{\gamma}{\tilde{\rho}_1 c H_1} , \quad H_1 = \frac{\tilde{p}_1}{g \tilde{\rho}_1} \left[ \exp\left(\frac{gh}{\tilde{p}_1/\tilde{\rho}_1}\right) - 1 \right] , \quad \lambda = \frac{g H_1}{c^2} ,$$

$$r = 1 + \left[ \exp\left(\frac{gh}{\tilde{p}_1/\tilde{\rho}_1}\right) - 1 \right]^{-1} , \quad M = r - 1 .$$

$\gamma$  can also be expressed in terms of the vertical distance

$\zeta$  at  $x = -\infty$  . For  $0 \leq \eta \leq 1$  ,



$$\gamma = \int_0^{\xi} \tilde{\rho}_{\infty} \tilde{u}_{\infty} dy = \frac{\tilde{p}_1^c}{g} \left[ \exp\left(\frac{g\tilde{p}_1^c h}{\tilde{p}_1}\right) - \exp\left(-\frac{g\tilde{p}_1^c}{\tilde{p}_1} (\xi-h)\right) \right],$$

and for  $1 \leq \eta \leq r$ ,

$$\begin{aligned} \gamma &= \frac{\tilde{p}_1^c}{g} \left[ \exp\left(\frac{g\tilde{p}_1^c h}{\tilde{p}_1}\right) - 1 \right] + \int_h^{\xi} \tilde{\rho}_{\infty} \tilde{u}_{\infty} dy \\ &= \frac{\tilde{p}_1^c}{g} \left[ \exp\left(\frac{g\tilde{p}_1^c h}{\tilde{p}_1}\right) - \exp\left(-\frac{g\tilde{p}_2^c}{\tilde{p}_1} (\xi-h)\right) \right]. \end{aligned}$$

## 5. Properties of the Solitary Waves

### Near the Two Critical Speeds

As mentioned in the Introduction, it can be seen from the complicated coefficients of the solitary wave equation that a complete analytical approach to the study of the solution for the solitary waves is almost impossible. However, we shall try to explore analytically the properties of the solution as much as we can with the help of numerical calculations. To begin with we first indicate some interesting properties of the critical speed  $\ell$ . From (4.8) we let

$$\begin{aligned} \ell_{+M} &= \frac{\alpha r + [(\alpha r)^2 - 4r(\alpha-1)]^{1/2}}{2(\alpha-1)}, \\ (5.1) \quad \ell_{-M} &= \frac{\alpha r - [(\alpha r)^2 - 4r(\alpha-1)]^{1/2}}{2(\alpha-1)} \end{aligned}$$



where

$$\infty > \alpha = \frac{T_2}{T_1} > 1$$

$$\infty > M = r - 1 = [\exp(\frac{gh}{\tilde{p}_0/\tilde{p}_0}) - 1]^{-1} > 0 .$$

We claim that  $\infty > \ell_{+M} > 1$  and  $1 > \ell_{-M} > 0$  . This can be shown as follows: rewrite the expression  $(\alpha r)^2 - 4r(\alpha-1)$  in the square root of (5.1) as  $(\alpha r - 2)^2 + 4(r-1)$  ; hence,

$$(\alpha r)^2 > (\alpha r)^2 - 4r(\alpha-1) > 0 .$$

This shows that  $\ell_{-M} > 0$  . Now let

$$z = 1 - \ell_M .$$

The quadratic equation for  $\ell$

$$(\alpha-1)(\ell M)^2 - \alpha r(\ell M) + r = 0$$

becomes

$$(\alpha-1)(1-z)^2 - \alpha r(1-z) + r = 0 ,$$

$$\text{i.e.} \quad (\alpha-1)z^2 + z[-2(\alpha-1) + \alpha r] + (\alpha-1)(1-r) = 0 .$$

However,  $\alpha > 1$  ,  $r > 1$  , the two roots of  $z$  must be of opposite signs, i.e.

$$z_{+} = 1 - \ell_{+M} < 0 , \quad z_{-} = 1 - \ell_{-M} > 0 ,$$

since  $\ell_{M+} > \ell_{M-}$  . This completes the proof.

Next we shall investigate certain limiting values of  $\ell$  .



Assume that  $M$  is positive finite. Then,

as  $\alpha \rightarrow 1^+$ ,

$$\ell_+ M \rightarrow +\infty, \quad \ell_- M \rightarrow 1^-;$$

as  $\alpha \rightarrow \infty$ ,

$$\ell_+ M \rightarrow r^+, \quad \ell_- M \rightarrow 0^+.$$

Now assume that  $\infty > \alpha > 1$ . We have,

as  $r \rightarrow 1^+$ ,

$$\ell_+ M \rightarrow \frac{1}{\alpha-1}^+, \quad \text{if } \alpha < 2,$$

$$\rightarrow 1^+, \quad \text{if } \alpha \geq 2;$$

$$\ell_- M \rightarrow 1^-, \quad \text{if } \alpha \leq 2,$$

$$\rightarrow \frac{1}{\alpha-1}^-, \quad \text{if } \alpha > 2;$$

as  $r \rightarrow \infty$ ,

$$\ell_+ M \rightarrow \infty, \quad \ell_- M \rightarrow \frac{1}{\alpha}^-.$$

With the knowledge of the properties of the critical speeds, we proceed to investigate the coefficients of the solitary wave equation given in (4.19). It is seen that the solution of equation (4.19) with any one of  $m_0$ ,  $m_1$  and  $m_2$  vanishing will be identically equal to zero if the conditions  $a_1(-\infty) = a_1'(-\infty) = a_1''(-\infty)$  are imposed. Therefore, we must determine along which curves in the domain



$\alpha > 1$  ,  $r > 1$  those coefficients will vanish. We take up  $m_2$  first which is given by

$$m_2 = \frac{M^2 r}{\ell M} \left[ \frac{2}{\ell M} - (2+\alpha) \right] .$$

Let  $m_2 = 0$  and assume that  $1 < r < \infty$  and  $1 < \alpha < \infty$  .

Then we have

$$\ell M = \frac{2}{\alpha+2} .$$

Substitution of the above equation for  $\ell M$  in

$$(5.2) \quad (\alpha-1)(\ell M)^2 - \alpha r(\ell M) + r = 0$$

yields

$$r = \frac{4(\alpha-1)}{\alpha^2 - 4} .$$

The portion of the locus of the above equation in the domain  $\alpha > 1$  ,  $r > 1$  is shown in Fig. 3; and  $\ell = \ell_-$  along the curve, below which  $m_2 < 0$  , and above which  $m_2 > 0$  . For the values of  $\ell_+$  ,  $m_2$  is always negative.

The coefficient of  $a_1'''(\sigma)$  given by (4.19) is

$$m_0 = \frac{M^3(1 - \ell M)}{(\ell M)^3} \left[ 2(1-r)r(1 - \alpha \ell M) \log \frac{r}{r-1} + 1 + r \right. \\ \left. + \ell M(1 - 2\alpha r) \right] .$$

For  $\infty > r > 1$  ,  $\infty > \alpha > 1$  ,  $\frac{M^3(1 - \ell M)}{(\ell M)^3} \neq 0$  and

$m_0 = 0$  implies

$$2(1-r)r(1 - \alpha \ell M) \log \frac{r}{r-1} + 1 + r + \ell M(1 - 2\alpha r) = 0 .$$



In combination with (5.2) we find that

$$\ell M = \frac{[2r(r-1)\log \frac{r}{r-1} - 1] \pm \left\{ [-2r(r-1)\log \frac{r}{r-1} + 1]^2 - 4r(r-1)[2r(r-1)\log \frac{r}{r-1} - 2r + 1] \right\}^{1/2}}{2r(r-1)\log \frac{r}{r-1} - 2r + 1},$$

$$\alpha = \frac{(\ell M)^2 - r}{(\ell M)^2 - r(\ell M)} = \frac{2r(1-r)\log \frac{r}{r-1} + 1 + r + \ell M}{2\ell M r[(1-r)\log \frac{r}{r-1} + 1]}.$$

Of course, we may eliminate  $\ell M$  and obtain an expression for  $\alpha$  in terms of  $r$ , but it is much more involved than the relation between  $\ell M$  and  $r$ . There are two curves found in the domain  $r > 1$ ,  $\alpha > 1$  along which  $m_0 = 0$  (Fig. 3). One corresponds to  $\ell_+$ , and the other,  $\ell_-$ . Let us denote the former by  $\ell_-$ -curve and the latter by  $\ell_+$ -curve. Then it is shown that  $m_0 > 0$  below the  $\ell_-$ -curve and to the left of the  $\ell_+$ -curve, and  $m_0 < 0$  above the  $\ell_-$ -curve and to the right of the  $\ell_+$ -curve.

Finally, we are confronting the coefficient of  $a_1(\sigma)a_1(\sigma)$ ,  $m_1$ , which can be rewritten in the form, for  $\infty > r > 1$ ,  $\infty > \alpha > 1$ ,

$$m_1 = (r-1) \left\{ 4r(r-1)\log \frac{r}{r-1} [\alpha((\ell M)^2 - (\ell M) + 1 - (\ell M)^2) + r^2\alpha[-2(\ell M) + 1 + 4(\ell M)^{-1}] + \alpha r[-2(\ell M)^2 + 6(\ell M) - 5 - 3(\ell M)^{-1}] + r^2[4 - 3(\ell M)^{-1} - 6(\ell M)^{-2}] + r[2(\ell M)^2 - (\ell M) - 10 + 12(\ell M)^{-1} + 3(\ell M)^{-2}] + \alpha \ell M + (\ell M + 4) \right\}$$



where, by (5.2)

$$\alpha = \frac{(\ell M)^2 - r}{(\ell M)(\ell M - 2)} .$$

We find by numerical calculations that  $m_1 > 0$  for  $\ell = \ell_+$  and there is a  $\ell_-$ -curve in  $\alpha > 1$ ,  $r > 1$  below which  $m_1 > 0$  and above which  $m_1 < 0$ .

Let us denote the six subdomains in  $\gamma > 1$ ,  $\alpha > 1$ , divided by the three  $\ell_-$ -curves by  $D_-$ -I,  $D_-$ -II, ...,  $D_-$ -VI and the two subdomains by the only  $\ell_+$ -curve by  $D_+$ -I,  $D_+$ -II (Fig. 3). We collect our results as below:

Domain	$m_0$	$m_1$	$m_2$	$\ell - \lambda^{(1)}$	Wave Type
$D_-$ -I	+	+	-	+	D
$D_-$ -II	+	-	-	-	D
$D_-$ -III	-	+	-	+	D
$D_-$ -IV	-	-	-	-	D
$D_-$ -V	-	+	+	-	D
$D_-$ -IV	-	-	+	+	D
$D_+$ -I	+	+	-	+	E
$D_+$ -II	-	+	-	+	E

In order to determine whether the solitary wave is of elevation or depression type, we must go back to (4.22). It is seen from the expressions for  $\bar{f}$  and  $\bar{F}$  that for given  $M$  and  $r$  the wave amplitude increases as  $\eta$  increases and reaches a maximum

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(1) The sign of  $\ell - \lambda$  must be such that the expression  
 $-(\ell - \lambda) \frac{m_2}{m_1}$  is always positive.



at  $\eta = 1$  , then stays at constant value for  $1 \leq \eta < r$  .

The wave type is determined by the sign of the expression

$$\frac{(1 - M\ell)(\ell - \lambda)m_2}{m_1} .$$

We denote by E the wave of elevation

and by D the wave of depression, and the last column in the above table indicates the wave type in each subdomain.

Now let us consider the limiting case that M is positive finite but  $\alpha \rightarrow 1^+$  . Then

$$(\ell_{+M}) \rightarrow \infty , \quad (\ell_{-M}) \rightarrow 1^- .$$

The latter is the only case we need to consider.

As  $(\ell_{-M}) \rightarrow 1^-$  ,

$$m_0 \rightarrow 0 ,$$

$$m_1 \rightarrow (r-1)[-2r^2 + 2r + 6]$$

$$m_2 \rightarrow -r(r-1)^2 ,$$

hence there exists no solitary wave solution. This limiting case, in fact, shows that as the temperature of the upper layer and lower layer tend to the same value the solitary wave will disappear as we have observed before.



Part III. Compressible Media of Infinite Depth  
with Non-Uniform Velocity Distribution at Equilibrium

1. Introduction

In Part I and Part II we are concerned with the situation that a solitary wave is generated by some disturbance in a compressible medium initially at rest. In reality this is usually not the case. In the equilibrium state the atmosphere may move with non-uniform velocity distribution. The work done here is to investigate the solitary waves in a compressible medium initially with arbitrary velocity profile. For simplicity, we only consider a medium of infinite depth at constant temperature. Nevertheless, the method employed here can be extended to cases of multi-layer polytropic or isothermal compressible media of finite or infinite depth without much difficulty.

The formulation of the problem is presented in Section 2. In Section 3 the linear theory predicts a critical speed for an arbitrary equilibrium velocity distribution. We treat the problem by means of the nonlinear theory in Section 4 as we did in Part I and Part II. The solitary wave solution is computed based on a velocity profile of exponential growth. It is confirmed that there exists no solitary solution as the velocity profile tends to a uniform one.



## 2. Formulation of the Problem

We consider a compressible medium at constant temperature, which fills up the whole upper half space. The medium is supported by a rigid plane bottom and the pressure  $p$  is assumed to be zero at infinity. There are no geometric constraints. In the equilibrium state the velocity profile as a function of the vertical distance only is assumed to be given. A cross section of the medium is the upper half plane (Fig. 4). It is assumed that a wave of permanent type moving to the left with constant velocity  $c$  has been generated in the medium due to some disturbance. We choose a coordinate system moving with the wave such that the  $x$ -axis coincides with the bottom and the  $y$ -axis passes through the crest or the trough of the wave and is positive upward (Fig. 4). With respect to the coordinate system the wave is stationary and the velocity of the medium at infinity in the equilibrium state is denoted by  $\tilde{u}_\infty(y)$ .

The governing equations are

$$\frac{\partial(\tilde{\rho}\tilde{u})}{\partial x} + \frac{\partial(\tilde{\rho}\tilde{v})}{\partial y} = 0 ,$$

$$\tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} = - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial x} ,$$

$$\tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial y} = - g - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial y} ,$$



$$\frac{\tilde{p}}{\tilde{p}_0} = \frac{\tilde{p}}{\tilde{p}_0} ,$$

where  $\tilde{\rho}(x,y)$  is the density,  $\tilde{u}(x,y)$ ,  $\tilde{v}(x,y)$  are the horizontal and vertical velocity components,  $\tilde{p}(x,y)$  is the pressure,  $g$  is the gravitational constant, and  $\tilde{p}_0$ ,  $\tilde{\rho}_0$  are respectively the reference pressure and density which may be taken as the ground level values in the equilibrium state.

Denote by  $\tilde{\psi}(x,y)$  the stream function such that

$$\psi_y = \tilde{\rho}\tilde{u} , \quad \psi_x = -\tilde{\rho}\tilde{v} .$$

The mass flux across any vertical plane from  $y = 0$  to  $y = \infty$  per unit breadth is given by

$$\tilde{\psi}(x,\infty) - \tilde{\psi}(x,0) = \int_0^\infty \tilde{\rho}_\infty \tilde{u}_\infty dy = \gamma_\infty$$

where we may set  $\tilde{\psi}(x,0) = 0$ .

Let

$$\tilde{\psi}(x,y) = \gamma$$

where  $0 \leq y < \infty$ ,  $0 \leq \gamma < \gamma_\infty$ , and assume that for any value of  $\gamma$  in  $0 \leq \gamma < \gamma_\infty$  there exists a unique solution of the above equation for  $y$  such that

$$y = \bar{f}(x,\gamma) .$$

Hereafter the bar notation is used to denote a quantity as a function of  $x$  and  $\gamma$ . If we choose  $\bar{f}$ ,  $\bar{p}$ ,  $\bar{\rho}$ , and  $\bar{u}$  as dependent variables then we have, as obtained in Part I, for  $0 \leq \gamma < \gamma_\infty$ ,  $-\infty < x < \infty$ ,



$$\begin{aligned}
\bar{u}_x &= -\bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma , \\
\bar{u} \bar{f}_{xx} + \bar{u}_x \bar{f}_x &= -\bar{\rho} g \bar{f}_\gamma - \bar{p}_\gamma , \\
(2.1) \quad \bar{\rho} \bar{u} \bar{f}_\gamma &= 1 , \\
\frac{\bar{p}}{\bar{\rho}_0} &= \frac{\bar{p}}{\bar{\rho}_0} ,
\end{aligned}$$

subject to the boundary conditions

$$\bar{f}(x,0) = 0 , \quad \bar{p}(x,\infty) = 0 .$$

In order to make  $\tilde{u}_\infty$  and  $\tilde{\rho}_\infty$  appear explicitly in the above equations it will be more convenient to use the vertical distance  $\zeta$  of the stream lines at  $x = -\infty$  as an independent variable. The relation between  $\gamma$  and  $\zeta$  is given by

$$(2.2) \quad \gamma = \int_0^\zeta \tilde{\rho}_\infty \tilde{u}_\infty dy ,$$

and it follows that

$$(2.3) \quad \frac{\partial \bar{\phi}}{\partial \gamma} = \frac{1}{\hat{G}_\infty} \frac{\partial \hat{\phi}}{\partial \zeta} ,$$

where

$$\hat{G}_\infty = \tilde{\rho}_\infty(\zeta) \tilde{u}_\infty(\zeta) ,$$

and the hat notation denotes a function of  $x, \zeta$ . However, if the transformation defined by (2.2), (2.3) is always possible, we must impose a condition on  $\hat{G}$ , i.e.  $\hat{G}$  is non-zero. Since  $\tilde{\rho}_\infty(\zeta)$  is always positive for  $0 \leq \zeta < \infty$ ,



we assume that

$$\tilde{u}_{\infty}(\zeta) \neq 0$$

for  $0 \leq \zeta \leq \infty$ .

By using  $x, \zeta$  as independent variables (2.1) becomes,  
for  $0 \leq \zeta < \infty$ ,  $-\infty < x < \infty$ ,

$$\begin{aligned} \hat{G}_{\infty} \hat{u}_x &= - \hat{f}_{\zeta} \hat{p}_x + \hat{f}_x \hat{p}_{\zeta}, \\ \hat{G}_{\infty} (\hat{u} \hat{f}_{xx} + \hat{u}_x \hat{f}_x) &= - \hat{\rho} g \hat{f}_{\zeta} - \hat{p}_{\zeta}, \\ (2.4) \quad \hat{\rho} \hat{u} \hat{f}_{\zeta} &= \hat{G}_{\infty}, \\ \frac{\hat{p}}{\hat{\rho}_0} &= \frac{\hat{\phi}}{\hat{\rho}_0}, \end{aligned}$$

subject to the boundary conditions

$$\hat{f}(x, 0) = 0, \quad \hat{p}(x, \infty) = 0.$$

In order to non-dimensionalize the above equations we introduce the following dimensionless variables

$$\begin{aligned} \xi &= \frac{x}{h}, & v &= \frac{\hat{v}}{c}, & u &= \frac{\hat{u}}{c}, \\ \eta &= \frac{\zeta}{h}, & p &= \frac{\hat{p}}{\hat{\rho}_0 c^2}, & \rho &= \frac{\hat{\rho}}{\hat{\rho}_0}, \\ f &= \frac{\hat{f}}{h}, & \lambda &= \frac{gh}{c^2}, & G_{\infty} &= \frac{\hat{G}_{\infty}}{\hat{\rho}_0 c}, \\ \rho_{\infty} &= \frac{\hat{\rho}_{\infty}}{\hat{\rho}_0}, & u_{\infty} &= \frac{\hat{u}_{\infty}}{c}, \end{aligned}$$



where  $h = \frac{\tilde{p}_0}{g\tilde{\rho}_0}$ . Then we have, from (2.4),

for  $0 < \eta < \infty$ ,  $-\infty < \sigma < \infty$ ,

$$G_\infty u_\xi = -f_\eta p_\xi + f_\xi p_\eta,$$

$$G_\infty (u f_{\xi\xi} + u_\xi f_\xi) = -\rho \lambda f_\eta - p_\eta,$$

(2.5)

$$\rho u f_\eta = G_\infty$$

$$p = \lambda p,$$

with the boundary conditions

$$f(\xi, 0) = 0, \quad p(\xi, \infty) = 0.$$

These equations look quite similar to those in Part I and Part II. We expect that the same procedure given before may as well be applied to the present problem without introducing much difficulty. However, since  $u_\infty(\eta)$  is arbitrary, it is anticipated that the coefficients of the solitary wave solution will be much more involved than before.

### 3. Linear Theory. Critical Speed

We first recall that in the equilibrium state the density distribution for an isothermal medium of infinite depth is of exponential decay, i.e.,



$$\tilde{\rho}_{\infty}(y) = \tilde{\rho}_0 \exp(-y/h) .$$

In terms of  $\rho_{\infty}$  and  $\eta$  , we have

$$(3.1) \quad \rho_{\infty}(\eta) = \exp(-\eta) .$$

The steady state solution for a parallel flow with velocity profile  $u_0 = u_{\infty}(\eta)$  is found from (2.5) and (3.1) as follows:

$$\begin{aligned} u_0 &= u_{\infty}(\eta) , \\ p_0 &= \lambda \rho_{\infty}(\eta) , \\ (3.2) \quad \rho_0 &= \rho_{\infty}(\eta) , \\ f_0 &= \eta . \end{aligned}$$

Suppose that a small disturbance is superposed on the parallel flow and we may write

$$\begin{aligned} u &= u_{\infty} + u^* , & p &= \lambda \rho_{\infty} + p^* \\ \rho &= \rho_{\infty} + \rho^* , & f &= \eta + f^* . \end{aligned}$$

If we substitute the above quantities in (2.5) and assume that any term which contains second order products of the starred variables can be neglected, we have the following linearized equations:

$$\begin{aligned} G_{\infty} u_{\xi}^* &= - p_{\xi}^* - \lambda \rho_{\infty} f_{\xi}^* , \\ G_{\infty} u_{\infty} f_{\xi \xi}^* &= - \lambda (\rho_{\infty} f_{\eta}^* + \rho^*) - p_{\eta}^* , \end{aligned}$$



$$(3.3) \quad \rho^* u_\infty + \rho_\infty u^* + G_\infty f_\eta^* = 0 ,$$

$$p^* = \lambda \rho^* ,$$

subject to the boundary conditions

$$f^*(\xi, 0) = 0 , \quad p^*(\xi, \infty) = 0 .$$

It is obtained from (3.3) that

$$\begin{aligned} G_\infty u_\infty f_{\xi\xi\xi}^* &= \lambda(u_\infty^2 - \lambda)^{-1} \rho_\infty u_\infty^2 f_{\xi\eta\eta}^* \\ &+ (u_\infty^2 - \lambda)^{-1} \rho_\infty [(u_\infty^2)_\eta (\lambda - \frac{\lambda u_\infty^2}{u_\infty^2 - \lambda}) - \lambda u_\infty^2] f_{\xi\eta}^* \\ &+ [(u_\infty^2 - \lambda)^{-2} (u_\infty^2)_\eta \lambda^2 \rho_\infty] f_\xi^* = 0 , \end{aligned}$$

and

$$\rho^* = (u_\infty^2 - \lambda)^{-1} [\lambda \rho_\infty f_\xi^* - \rho_\infty u_\infty^2 f_{\xi\eta}^*] .$$

We first try to find a solution for  $f_\xi^*$ . Let

$$f_\xi^* = H(\xi)F(\eta) ,$$

and from (3.4) we obtain

$$\begin{aligned} H_{\xi\xi\xi} + v^2 H &= 0 , \\ F_{\eta\eta} + [- (u_\infty^2 - \lambda)^{-1} (u_\infty^2)_\eta + (\rho_\infty u_\infty^2)^{-1} (\rho_\infty u_\infty^2)_\eta] F_\eta \\ &+ (u_\infty^2 - \lambda)^{-1} (u_\infty^2)^{-1} \lambda (u_\infty^2)_\eta F = - v^2 \lambda^{-1} (u_\infty^2 - \lambda) F . \end{aligned}$$

The solution for  $H$  is

$$H = A \cos(v\xi + B)$$



where  $a$  and  $b$  are two arbitrary constants. Since the critical speed  $\ell$  is defined as

$$\ell = \lim_{v \rightarrow 0} \lambda(v), \quad (1)$$

the following asymptotic method is suggested to find the value of  $\ell$  while a general discussion of the solution of  $F$  will be given in the Appendix. Let us assume that, for small values of  $v$ ,

$$\lambda = \ell + v^2 \lambda_1 + v^4 \lambda_2 + \dots,$$

$$F = F_0(\eta) + v^2 F_1(\eta) + v^4 F_2(\eta) + \dots.$$

The equation for  $F_0(\eta)$  is found as

$$\begin{aligned} & F_{0\eta\eta} + \{ - (u_\infty^2 - \ell)^{-1} (u_\infty^2)_\eta + (\rho_\infty u_\infty^2)^{-1} (\rho_\infty u_\infty^2)_\eta \} F_{0\eta} \\ (3.4) \quad & + (u_\infty^2 - \ell)^{-1} (u_\infty^2)^{-1} \ell (u_\infty^2)_\eta F_0 = 0, \end{aligned}$$

subject to

$$F(0) = 0,$$

$$\lim_{\eta \rightarrow \infty} (u_\infty^2 - \ell)^{-1} [\ell \rho_\infty F_0 - \rho_\infty u_\infty^2 F_{0\eta}] = 0.$$

The general solution for  $F_0$  is obtained as follows:

$$F_0 = C[-1 + \ell e^\eta \int_\eta^\infty e^{-\eta'} u_\infty^{-2}(\eta') d\eta'] + D e^\eta.$$

---

(1) For definition of the critical speed, c.f. [1].



The condition at  $\eta = \infty$  is no more than a boundedness condition for  $F$  and requires  $D = 0$ . Since  $F_0(0) = 0$ , if we assume that the motion is other than a parallel flow, i.e.  $C \neq 0$ , we obtain the critical speed

$$\ell = \left[ \int_0^{\infty} e^{-\eta'} u_{\infty}^{-2}(\eta') d\eta' \right]^{-1}.$$

It is interesting to consider the case  $u_{\infty} = \text{const}$ .

$$(1) \quad u_{\infty}^2 \neq \lambda.$$

From (3.5) we have

$$F_{\eta\eta} - F_{\eta} = -v^2 \lambda^{-1} (u_{\infty}^2 - \lambda) F.$$

The solution for  $F$  which satisfies  $F(0) = 0$  is

$$F = c(e^{m_1 \eta} - e^{m_2 \eta})$$

where  $c$  is an arbitrary constant and

$$m_1 = \frac{1}{2} [1 - \sqrt{1 - 4v^2(u_{\infty}^2 - \lambda)\lambda^{-1}}],$$

$$m_2 = \frac{1}{2} [1 + \sqrt{1 - 4v^2(u_{\infty}^2 - \lambda)\lambda^{-1}}].$$

We may use the condition  $\rho_{\xi}^*(\xi, \infty) = 0$  to determine  $\lambda$  and consider the cases  $u_{\infty}^2 > \lambda$  and  $u_{\infty}^2 < \lambda$ . However  $F$  is always unbounded as  $\eta \rightarrow \infty$  except that  $c = 0$ . Hence for  $u_{\infty}^2 \neq \lambda$  linear theory fails or the solution for  $F$  is a trivial one.

$$(2) \quad u_{\infty}^2 = \lambda.$$

From the equation for  $\rho_{\xi}^*$  in (3.5) we have



$$F_{\eta} - F = 0 .$$

It is seen that  $F \equiv 0$  if  $F(0) = 0$  . The above results, in fact, are exactly what we obtained from the linearized equations for the case  $\eta = 1$  in Part I, except that a different independent variable is used in each case.

#### 4. Nonlinear Theory. Solitary Wave Solution

It is assumed that a solitary wave moves with a speed such that  $\lambda = \frac{gh}{c^2}$  is near some positive value  $\ell$  , which is to be determined later. The equations (2.5) can be written as,  
for  $0 < \eta < \infty$  ,  $-\infty < \xi < \infty$  ,

$$G_{\infty} u_{\xi} = - f_{\eta} p_{\xi} + f_{\xi} p_{\eta} ,$$

$$G_{\infty} (u f_{\xi \xi} + u_{\xi} f_{\xi}) = (\ell - \lambda) \rho f_{\eta} - \ell \rho f_{\eta} - p_{\eta} ,$$

(4.1)

$$\rho u f_{\eta} = G_{\infty} ,$$

$$p = - (\ell - \lambda) \rho + \ell \rho ,$$

subject to the boundary conditions

$$f(\xi, 0) = 0 , \quad p(\xi, \infty) = 0 .$$

We let

$$\varepsilon = \ell - \lambda$$

and introduce a new variable



$$\sigma = \xi \sqrt{\varepsilon} .$$

Then (4.1) becomes,

for  $0 < \eta < \infty$  ,  $-\infty < \sigma < \infty$  ,

$$G_{\infty} u_{\sigma} = - f_{\eta} p_{\sigma} + f_{\sigma} p_{\eta} ,$$

$$\varepsilon G_{\infty} (u f_{\sigma\sigma} + u_{\sigma} f_{\sigma}) = \varepsilon \rho f_{\eta} - \ell \rho f_{\eta} - p_{\eta} ,$$

(4.2)

$$\rho u f_{\eta} = G_{\infty} ,$$

$$p = - \varepsilon \rho + \ell \rho ,$$

together with the boundary conditions

$$f(\sigma, 0) = 0 , \quad p(\sigma, \infty) = 0 .$$

As before, suppose that all the dependent variables can be expanded in integral powers of  $\varepsilon$  , i.e.

$$(4.3) \quad \phi(\sigma, \eta, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \phi_k(\sigma, \eta)$$

where  $\phi$  stands for  $p$  ,  $\rho$  ,  $u$  and  $f$  . Substitution of (4.3) in (4.2) will yield a sequence of equations and boundary conditions for the successive approximations by letting the coefficients of like powers of  $\varepsilon$  be equal.

The equations for the zero-th order approximation are, for  $0 < \eta < \infty$  ,  $-\infty < \sigma < \infty$  ,

$$G_{\infty} u_{0\sigma} = - p_{0\sigma} f_{0\eta} + f_{0\sigma} p_{0\eta} ,$$

$$0 = - \ell \rho_0 f_{0\eta} - p_{0\eta} ,$$

(4.4)



$$\rho_0 u_0 f_{0\eta} = G_\infty ,$$

$$p_0 = \ell \rho_0 ,$$

with

$$f_0(\sigma, 0) = 0 , \quad p(\sigma, \infty) = 0 .$$

Let us assume that  $u_0 = u_\infty(\eta)$  which represents a steady parallel flow. The solution for the zero-th order approximation is found as:

$$u_0 = u_\infty(\eta)$$

$$p_0 = \ell \rho_\infty(\eta)$$

(4.5)

$$\rho_0 = \rho_\infty(\eta)$$

$$f_0 = \eta .$$

The equations for the first order approximation are,  
for  $0 < \eta < \infty$  ,  $-\infty < \sigma < \infty$  ,

$$G_\infty u_{1\sigma} = - f_{0\eta} p_{1\sigma} + f_{1\sigma} p_{0\eta} ,$$

$$0 = \rho_0 f_{0\eta} - \ell(\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - p_{1\eta} ,$$

(4.6)

$$\rho_1 u_0 f_{0\eta} + \rho_0 u_1 f_{0\eta} + \rho_0 u_0 f_{1\eta} = 0 ,$$

$$p_1 = \ell \rho_1 - \rho_0 ,$$

with the boundary conditions

$$f_1(\sigma, 0) = 0 , \quad p_1(\sigma, \infty) = 0 .$$

It is obtained from (4.6) that



$$(u_{\infty}^2 p_{1\eta\sigma})_{,\eta} = \frac{\rho_{\infty}'}{\rho_{\infty}} (u_{\infty}' p_{1\eta\sigma}) .$$

Integration of the above equation and making use of  $p_1(\sigma, \infty) = 0$  yields

$$p_{1\sigma} = a'(\sigma) \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' ,$$

where  $a'(\sigma)$  is an arbitrary function of  $\sigma$ . Since the flow is assumed to reach the equilibrium state at  $x = -\infty$ , i.e.  $a(\sigma) \rightarrow 0$  as  $\sigma \rightarrow -\infty$ , we obtain

$$p_1 = a(\sigma) \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' ,$$

where we assume that  $a(\sigma)$  is not identically equal to zero. It follows from (4.6) and the solution for  $p_1$  that

$$\begin{aligned} f_{1\sigma} &= (p_{0\eta})^{-1} [\ell^{-1} u_{\infty}^2 p_{1\eta\sigma} + f_{0\eta} p_{1\sigma}] \\ &= -\ell^{-1} \rho_{\infty}^{-1} a'(\sigma) [-\ell^{-1} \rho_{\infty} + \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta'] . \end{aligned}$$

By the equilibrium condition at  $x = -\infty$ , we have

$$f_1 = -\ell^{-1} \rho_{\infty}^{-1} a(\sigma) [-\ell^{-1} \rho_{\infty} + \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta'] .$$

Since  $f_1(\sigma, 0) = 0$ , and  $a(\sigma) \neq 0$ , it follows that

$$-1 + \ell^{-1} \int_0^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' = 0 ,$$



i.e. 
$$\ell = \left[ \int_0^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta' \right]^{-1}.$$

The value for  $\ell$  confirms the critical speed we have found by the linear theory.

In summary, the solutions for the first order approximation are

$$\begin{aligned} p_1 &= a(\sigma) \int_\eta^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta', \\ \rho_1 &= \frac{1}{\ell} \left[ a(\sigma) \int_\eta^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta' + \rho_\infty \right] \\ (4.7) \quad u_1 &= \frac{1}{\ell} \left[ -\frac{a(\sigma)}{u_\infty} - u_\infty \right], \\ f_1 &= a(\sigma) \left[ \ell^{-2} - \frac{1}{\ell \rho_\infty} \int_\eta^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta' \right]. \end{aligned}$$

To determine the function  $a(\sigma)$  we must go one step further. The equations for the second order approximation are, for  $0 < \eta < \infty$ ,  $-\infty < \sigma < \infty$ ,

$$\begin{aligned} G_\infty u_{2\sigma} &= - (p_{2\sigma} f_{0\eta} + p_{1\sigma} f_{1\eta}) + (f_{2\sigma} p_{0\eta} + f_{1\sigma} p_{1\eta}), \\ G_\infty (u_\infty f_{1\sigma\sigma}) &= (\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - \ell (\rho_0 f_{2\eta} + \rho_1 f_{1\eta} + \rho_2 f_{0\eta}) - p_{2\eta}, \\ \rho_0 u_{2\sigma} f_{0\eta} + \rho_0 u_{0\sigma} f_{2\eta} + \rho_2 u_{0\sigma} f_{0\eta} + \rho_0 u_{1\sigma} f_{1\eta} + \rho_1 u_{0\sigma} f_{1\eta} + \rho_1 u_{1\sigma} f_{0\eta} &= 0, \\ (4.8) \quad p_2 &= \ell \rho_2 - \rho_1, \end{aligned}$$



subject to the boundary conditions,

$$f_2(\sigma, 0) = 0, \quad p_2(\sigma, \infty) = 0.$$

It is obtained from (4.8) that

$$(u_\infty^2 p_{2\eta\sigma})_\eta - \frac{\rho_\infty'}{\rho_\infty} u_\infty^2 p_{2\sigma\eta} = g_4(\sigma, \eta),$$

$$f_{2\sigma} = p_{0\eta}^{-1} [\ell^{-1} u_\infty^2 p_{2\eta\sigma} + p_{2\sigma} f_{0\eta} + \ell^{-1} g_{2\sigma} - g_1],$$

where

$$(4.9) \quad g_4(\sigma, \eta) = -g_{2\eta\sigma} - \rho_0^{-1} \rho_{1\sigma} p_{0\eta} f_{0\eta} + p_{0\eta} \rho^{-1} [-u_\infty^{-2} g_{2\sigma} \\ + \ell u_\infty^{-2} g_{3\sigma}] + p_{0\eta} p_{0\eta}^{-1} [g_{2\sigma} - \ell g_1] + \ell g_{1\eta},$$

$$g_1(\sigma, \eta) = -p_{1\sigma} f_{1\eta} + f_{1\sigma} p_{1\eta},$$

$$g_2(\sigma, \eta) = G_\infty u_\infty^3 f_{1\sigma\sigma} + G_\infty u_1 + \ell \rho_\infty u_1^2,$$

$$g_3(\sigma, \eta) = -u_\infty^2 \rho_1 f_{1\eta} + \rho_\infty u_1^2.$$

Now from the condition  $f_{1\sigma}(\sigma, 0) = 0$  and  $\ell = [\int_0^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta']^{-1}$  we obtain the following condition

$$(4.10) \quad u_\infty^2(0) p_{2\eta\sigma}(\sigma, 0) + p_{2\sigma}(\sigma, 0) [\int_0^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta']^{-1} + g_{2\sigma}(\sigma, 0) \\ - g_1(\sigma, 0) [\int_0^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta']^{-1} = 0.$$

The equation for  $p_{2\sigma}$  can also be rewritten as

$$(4.11) \quad \left( \frac{u_\infty^2 p_{2\sigma\eta}}{\rho_\infty} \right)_\eta = \frac{1}{\rho_\infty} g_4(\sigma, \eta).$$



Multiplying both sides of the above equation by  $\int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta'$

and then integrating with respect to  $\eta$  from 0 to  $\infty$ , we have

$$\begin{aligned} & - \int_0^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' u_{\infty}^2(0) p_{2\sigma\eta}(\sigma, 0) - p_{2\sigma}(\sigma, 0) \\ & = \int_0^{\infty} \left( \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' \right) \frac{g_4(\sigma, \eta)}{\rho_{\infty}(\eta)} d\eta. \end{aligned} \quad (1)$$

It follows from (4.10) that

$$\begin{aligned} & \int_0^{\infty} \left( \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' \right) \frac{g_4(\sigma, \eta)}{\rho_{\infty}(\eta)} d\eta \\ & = \int \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' g_{2\sigma}(\sigma, 0) - g_1(\sigma, 0). \end{aligned}$$

By some lengthy but straightforward calculations, from (4.9) and (4.11) we finally reach the equation

$$m_0 a''''(\sigma) + m_1 a'(\sigma) a(\sigma) + m_2 a'(\sigma) = 0$$

where

---

(1) In deriving this equation, we must assume the condition

$$\lim_{\eta \rightarrow \infty} \left( \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' \right) \frac{u_{\infty}^2 p_{2\sigma\eta}}{\rho_{\infty}} = 0.$$

However, from the equation (4.11) if  $u_{\infty}^2 \not\rightarrow 0$  as  $\eta \rightarrow \infty$ , we find that  $g_4(\sigma, \eta) = O(e^{-\eta})$  and  $p_{2\sigma\eta} \rightarrow 0$  as  $\eta \rightarrow 0$ , and the above condition follows immediately.



$$\begin{aligned}
m_0 &= - \int_0^\infty \rho_\infty^{-1} F(\eta) [\ell^{-2} G_\infty u_\infty^3 - \ell^{-1} u_\infty^4 F(\eta)]_\eta d\eta \\
&\quad + \int_0^\infty \rho_\infty^{-1} F(\eta) [\ell^{-2} G_\infty u_\infty^3 - \ell^{-1} u_\infty^4 F(\eta)]_\eta (\ell u_\infty^{-2} - 1) d\eta \\
&\quad - F(0) [\ell^{-2} G(0) u_\infty^3(0) - \ell^{-1} u_\infty^4(0) F(0)] , \\
m_1 &= + \int_0^\infty F(\eta) [-3\ell^{-1} \rho_\infty^{-1} (\frac{\rho_\infty}{u_\infty^2})_\eta + 2u_\infty^{-4} (1 - \ell^{-2}) - 3\ell^{-1} u_\infty^{-2} \\
&\quad - (2 + \rho_\infty^{-2}) F^2(\eta) + 4\rho_\infty^{-1} u_\infty^{-2} F(\eta) - \rho_\infty^{-1} (\rho_\infty^{-1} F^2(\eta) \\
&\quad - 2u_\infty^{-2} F(\eta))_\eta] d\eta - \ell^{-1} F^2(0) - \ell^{-2} u_\infty^{-2}(0) , \\
m_2 &= \int_0^\infty F(\eta) [-4u_\infty^{-2} - F(\eta)(\rho_\infty + \rho_\infty^{-1})] d\eta - \ell^{-1} F(0) , \\
F(\eta) &= \int_\eta^\infty \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta' .
\end{aligned}$$

Suppose that none of  $m_0$ ,  $m_1$  and  $m_2$  are equal to zero, and we also impose the conditions

$$a'(-\infty) = a''(-\infty) = 0 , \quad a'(0) = 0 ,$$

then the solution for  $a(\sigma)$  is

$$a(\sigma) = - \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{-\frac{m_2}{m_0}} .$$

Assume that the successive approximations up to the first order give a sufficiently accurate representation of a



solitary wave. We have, in terms of the independent variables  $x$  and  $\xi$ , for  $0 \leq \xi < \infty$ ,  $-\infty < x < \infty$ ,

$$\hat{p} \approx \tilde{\rho}_0 c^2 [\ell \rho_\infty - F(\eta)(\ell - \lambda) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{h} \sqrt{-\frac{m_2}{m_0}(\ell - \lambda)}],$$

$$\hat{\rho} \approx \tilde{\rho}_0 \rho_\infty (2 - \frac{\lambda}{\ell}) - \tilde{\rho}_0 (\ell - \lambda) F(\eta) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{h} \sqrt{-\frac{m_2}{m_0}(\ell - \lambda)},$$

$$\hat{u} \approx c \frac{\lambda}{\ell} u_\infty + c \ell^{-1} (\ell - \lambda) u_\infty^{-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{h} \sqrt{-\frac{m_2}{m_0}(\ell - \lambda)},$$

$$\hat{r} \approx h\eta - h(\ell - \lambda) [\ell^{-2} - \ell^{-1} \rho_\infty^{-1} F(\eta)] \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{h} \sqrt{-\frac{m_2}{m_0}(\ell - \lambda)},$$

$$\hat{v} \approx \frac{c\lambda}{\ell} [\ell^{-2} - \ell^{-1} \rho_\infty^{-1} F(\eta)] \frac{3m_2(\ell - \lambda)}{m_1} \left(-\frac{m_2}{m_0}(\ell - \lambda)\right)^{1/2} \operatorname{sech}^2 \frac{x}{h}$$

$$\sqrt{-\frac{m_2}{m_0}(\ell - \lambda)} \tanh \frac{x}{h} \sqrt{-\frac{m_2}{m_0}(\ell - \lambda)},$$

where  $\eta = \frac{\xi}{h}$ ,  $\rho_\infty = \exp(-\eta)$ ,  $h = \frac{\tilde{p}_0}{\tilde{\rho}_0 c}$ ,  $\lambda = \frac{gh}{c^2}$ .

### 5. An Example

In the following we shall give a concrete example based upon a special velocity profile

$$u_\infty = 1 - k \exp(-\eta), \quad 0 < k < 1.$$

The two extreme cases,  $k \rightarrow 0^+$  and  $k \rightarrow 1^-$ , will be



discussed at the end of the section. By repeated integrations, we find that

$$m_0 = \frac{-4k^2 - 5k - 5}{6(1-k)^2} ,$$

$$m_1 = \frac{1}{6k(1-k)^4} [k^5 + 6k^4 - 27k^3 + 32k^2 - 25k + 2] ,$$

$$m_2 = \frac{2k - 1}{2(1-k)^2} ,$$

$$\ell = (1-k) ,$$

where

$$m_0 < 0 , \quad \text{for } 0 < k < 1 ;$$

$$\begin{aligned} m_1 &> 0 , & \text{for } 0 < k < k_0 , \\ &= 0 , & \text{for } k = k_0 \approx 0.08 , \\ &< 0 , & \text{for } k_0 < k < 1 ; \end{aligned}$$

$$\begin{aligned} m_2 &> 0 , & \text{for } 1/2 < k < 1 , \\ &= 0 , & \text{for } k = 1/2 , \\ &< 0 , & \text{for } 0 < k < 1/2 . \end{aligned}$$

The expression for  $f$  is found as:

$$\hat{f} \approx \xi - h(\ell - \lambda) [1 - (1-k)(1 - ke^{-\eta})^{-1}] \frac{9k(2k - 1)}{N(k)} \operatorname{sech}^2 \frac{x}{h} \\ \sqrt{3(2k - 1)(4k^2 + 5k + 5)(\ell - \lambda)}$$

where

$$N(k) = k^5 + 6k^4 - 27k^3 + 32k^2 - 25k + 2 .$$



For a given  $k$ , the wave amplitude increases as  $\eta$  increases and finally reaches a finite value as  $\eta \rightarrow \infty$ . The wave type depends upon the sign of  $-(\ell-\lambda) \frac{m_2}{m_1}$ .

Let us divide the open interval  $0 < k < 1$  into three open intervals,  $I_1 = (0, k_0)$ ,  $I_2 = (k_0, 1/2)$ , and  $I_3 = (1/2, 1)$ . We list our results as follows:

$I$ ,	$m_0$ ,	$m_1$ ,	$m_2$ ,	$\ell-\lambda$ ,	Wave type
$I_1$	-	+	+	+	E
$I_2$	-	-	+	+	D
$I_3$	-	-	-	-	D

When  $k = k_0$ ,  $k = 1/2$ , there exists no solitary wave solution. It is also seen from the expression for  $\hat{f}$  that as  $k \rightarrow 0$ ,  $f_1 \rightarrow 0$ , and as  $k \rightarrow 1$ ,

$$\hat{f}_1 \rightarrow +(\ell-\lambda) \frac{9}{11} \sec^2 \frac{x}{h} \sqrt{42(\ell-\lambda)}, \text{ for } \eta > 0,$$

$$\hat{f}_1 \rightarrow 0, \text{ for } \eta = 0.$$

The former confirms that the solitary wave solution disappears for an isothermal layer of infinite depth as the velocity distribution tends to a uniform one. The latter, however, shows a very queer situation. The limiting solution of  $\hat{f}$  as  $k \rightarrow 1$  possesses a discontinuity at  $\eta = 0$ . This is physically impossible and the theory fails. In fact, in this case  $m_0$ ,  $m_1$  and  $m_2$  tend to  $+\infty$  or  $-\infty$  as  $k \rightarrow 1^-$ .



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# Appendix I. Solution of the Linearized Equations

in Part I.

We shall investigate the solution of the equation

$$(I.1) \quad \left[1 - \frac{1}{n\lambda} (1-\eta)^{\frac{1}{n}-1}\right] F_{\eta\eta} + \left[-\frac{2}{n} (1-\eta)^{-1} + \frac{1}{\lambda} \left(\frac{1}{n} + \frac{1}{n^2}\right) (1-\eta)^{\frac{1}{n}-2}\right] F_{\eta} \\ + \left[\left(\frac{1}{n^2} - \frac{1}{n}\right) (1-\eta)^{-2}\right] F = v^2 \left[(1-\eta)^{-\frac{1}{n}} - \frac{1}{n\lambda} (1-\eta)^{-1}\right]^2 F ,$$

subject to the boundary conditions

$$F(0) = 0 ,$$

$$\lim_{\eta \rightarrow 1^-} (1-\eta) \left[(1-\eta)^{1-\frac{1}{n}} - \frac{1}{n\lambda}\right]^{-1} \left[(1-\eta)^{\frac{1}{n}} F_{\eta} - \lambda F\right] = 0 .$$

In order to facilitate our discussion we let

$$z = (1-\eta)^{\frac{n-1}{n}} ,$$

and (I.1) becomes<sup>(1)</sup>

(I.2)

$$z \left(\frac{n-1}{n}\right)^2 \left(z - \frac{1}{n\lambda}\right) F_{zz} + \left(\frac{n-1}{n^2}\right) \left(z - \frac{1}{\lambda}\right) F_z + \left(\frac{1}{n^2} - \frac{1}{n}\right) F = v^2 \left(z - \frac{1}{n\lambda}\right)^2 F .$$

The two independent solutions of (I.2) for  $v = 0$  are found as

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(1) Hereafter we use the same capital letter  $F$  to denote a function of  $z$  .



$$F_1 = 1 - \lambda z ,$$

$$F_2 = z^{-\frac{1}{n-1}} .$$

Now we construct two integral equations

$$\bar{F}_1 = F_1 - \nu^2 \int_z^1 K(\sigma, z) \bar{F}_1(\sigma) d\sigma ,$$

(I.3)

$$\bar{F}_2 = F_2 - \nu^2 \int_z^1 K(\sigma, z) \bar{F}_2(\sigma) d\sigma ,$$

where

$$K(\sigma, z) = \frac{F_1(z)F_2(\sigma) - F_2(z)F_1(\sigma)}{F_1^1(\sigma)F_2(\sigma) - F_2^1(\sigma)F_1(\sigma)} \sigma^{-1} \left(\frac{n}{n-1}\right)^2 \left(\sigma - \frac{1}{n\lambda}\right)^2$$

$$= - \frac{n}{(n-1)\lambda} [(1 - \lambda z) - (1 - \lambda \sigma)(z^{-1}\sigma)^{\frac{1}{n-1}}] .$$

By the theory of Volterra integral equations it is easily shown that since  $F_1$  ,  $F_2$  ,  $K(\sigma, z)$  are bounded for

$0 < \eta_0 \leq \eta \leq 1$  ,  $\eta \leq \sigma \leq 1$  ,  $\bar{F}_1(z)$  and  $\bar{F}_2(z)$  are uniquely determined by (II.3) and also satisfy (I.2). From (I.3) we see that  $z = \frac{1}{n\lambda}$  is an apparent singularity of (I.2) for  $0 < \frac{1}{n\lambda} \leq 1$  . It is recalled that the equation for  $p_{\xi}^*$  is given by

$$p_{\xi}^* = G(\xi)(1-\eta)[(1-\eta)^{1-\frac{1}{n}} - \frac{1}{n\lambda}]^{-1}[(1-\eta)^{\frac{1}{n}}F_{\eta} - \lambda F]$$

(I.4)

$$= G(\xi)z^{\frac{n-1}{n}}(z - \frac{1}{n\lambda})^{-1}[-\frac{n-1}{n}Fz - \lambda F] .$$



We claim that if  $z = \frac{1}{n\lambda}$  is an apparent singularity of  $p_{\xi}^*$  for  $v = 0$  then it must be so for  $v \neq 0$ . First we note that

$$(I.5) \quad \begin{aligned} -\frac{n-1}{n} F_{1z} - \lambda F_1 &= \lambda^2 \left( z - \frac{1}{n\lambda} \right), \\ -\frac{n-1}{n} F_{2z} - \lambda F_2 &= -\lambda z^{-\frac{n}{n-1}} \left( z - \frac{1}{n\lambda} \right). \end{aligned}$$

Let

$$F = c_1 F_1 + c_2 F_2,$$

and substitute  $F$  in (I.4). It is seen that  $z = \frac{1}{n\lambda}$  is an apparent singularity of  $p_{\xi}^*$  as  $v = 0$  for  $0 < \frac{1}{n\lambda} \leq 1$ .

For  $v \neq 0$ , we rewrite  $K(\sigma, \eta)$  as

$$K(\sigma, \eta) = [F_1(z)F_2(\sigma) - F_2(z)F_1(\sigma)] \frac{n}{(n-1)\lambda} \sigma^{-\frac{1}{n-1}},$$

and note that

$$K(z, z) = 0.$$

Therefore, from (I.3) we have

$$\begin{aligned} \bar{F}_{iz} &= F_{iz} - v^2 \int_z^1 [F_{1z}(z)F_2(\sigma) - F_{2z}(z)F_1(\sigma)] \frac{n}{(n-1)} \sigma^{\frac{1}{n-1}} \\ &\quad \bar{F}_i(\sigma) d\sigma, \quad i = 1, 2. \end{aligned}$$

Let

$$F = c_1 \bar{F}_1 + c_2 \bar{F}_2,$$

where  $c_1(1-\lambda) + c_2 = 0$  since  $F(1) = 0$ .

Then from (I.4) it is obtained that



$$p_{\xi}^* = G(\xi) z^{\frac{n-1}{n}} (z - \frac{1}{n\lambda})^{-1} x$$

$$\left\{ c_1 \left( -\frac{n-1}{n} F_{1z}(z) - \lambda F_1(z) \right) + c_2 \left( -\frac{n-1}{n} F_{2z}(z) - \lambda F_2(z) \right) \right. \\ \left. - v^2 \int_z^1 \left[ \left( -\frac{n-1}{n} F_{1z}(z) - \lambda F_1(z) \right) F_2(\sigma) - \left( -\frac{n-1}{n} F_{2z}(z) - \lambda F_2(z) \right) F_1(\sigma) \right] \times \frac{n}{n-1} \sigma^{\frac{1}{n-1}} (c_1 \bar{F}_1(\sigma) + c_2 \bar{F}_2(\sigma)) d\sigma \right\}.$$

It follows from (I.5) that  $z = \frac{1}{n\lambda}$  for  $0 < \frac{1}{n\lambda} \leq 1$  is an apparent singularity of  $p_{\xi}^*$  for  $v \neq 0$ .

In the neighborhood of  $z = 0$ , we apply the method of Frobenius. Put

$$F = \sum_{m=0}^{\infty} a_m z^{c+m}.$$

Substitution of  $F$  in (I.2) yields

$$a_0 \left[ \left( \frac{n-1}{n} \right) c(c-1) + c \right] = 0.$$

This gives

$$c = 0 \quad \text{and} \quad c = -\frac{1}{n-1},$$

and we always obtain two solutions, one of which is bounded and the other of which may contain a logarithmic term and is always unbounded. It is easily shown that only the bounded solution satisfies the boundary condition as  $\eta \rightarrow 1^-$  given in (I.1). The recurrence formula for the series expansion



of this bounded solution is given by

$$a_{m+1} \left( \frac{n-1}{n} \right)^2 \frac{(m+1)}{\lambda} \left( \frac{m}{n} + \frac{1}{n-1} \right) = a_m \left[ \left( \frac{n-1}{n} \right)^2 m(m-1) \right. \\ \left. + \left( \frac{n-1}{n^2} \right) m + \left( \frac{1}{n^2} - \frac{1}{n} \right) - \frac{v^2}{n^2 \lambda^2} \right] - v^2 \left[ a_{m-2} + \frac{2}{n\lambda} a_{m-1} \right]$$

for  $m \geq 0$ , and  $a_{-2} = a_{-1} = 0$ , and

$$F = \sum_{m=0}^{\infty} a_m z^m$$

$$(I.6) \quad = a_0 [1 - \lambda z + v^2 F(z, k^2, \lambda)] .$$

It is also easily shown that the series expansion converges for  $|z| < \frac{1}{n\lambda}$ .

Finally, we shall verify whether the linearizing procedure is consistent. The terms we have neglected from the full equations (2.17) are

$$f_{\eta}^* p_{\xi}^* , f_{\xi}^* p_{\eta}^* , u^* f_{\xi\xi}^* , u_{\xi}^* f_{\xi}^* , \rho^* f_{\eta}^* , u^* \rho^* f_{0\eta} , u^* \rho_0 f_{\eta}^* , \\ \rho^* f_{\eta}^* , \rho^* u^* f_{\eta}^* .$$

Since in the neighborhood of  $\eta = 1$ ,

$$f_{\eta}^* = f_z^* \frac{dz}{d\eta} = O(1-\eta)^{-\frac{1}{n}} ,$$

$$p_{\xi}^* = O(1-\eta) ,$$



$$u^* = O(1) \ ,$$

$$f_{\xi}^* = O(1) \ ,$$

it is easily verified that all the terms listed above are bounded at  $\eta = 1$  and our linearizing procedure is then justified.



## Appendix II. Solution of the Linearized Equations

in Part III.

In the Appendix we shall consider the equation

$$\begin{aligned}
 & F_{\eta\eta} + [- (u_{\infty}^2 - \lambda)^{-1} (u_{\infty}^2)_{\eta} + (\rho_{\infty} u_{\infty}^2)^{-1} (\rho_{\infty} u_{\infty}^2)_{\eta}] F_{\eta} \\
 (II.1) \quad & + (u_{\infty}^2 - \lambda)^{-1} (u_{\infty}^2)^{-1} \lambda (u_{\infty}^2)_{\eta} F = - v^2 (u_{\infty}^2 - \lambda) \lambda^{-1} F
 \end{aligned}$$

subject to the boundary conditions

$$F(0) = 0 ,$$

$$\lim_{\eta \rightarrow \infty} (u_{\infty}^2 - \lambda)^{-1} [\lambda \rho_{\infty} F - \rho_{\infty} u_{\infty}^2 F_{\eta}] = 0$$

as given in (3.5) where  $u_{\infty} \neq \text{constant}$  . The two independent solutions of (1) as  $v = 0$  are found as

$$\begin{aligned}
 & F_1 = - \lambda^{-1} + \rho_{\infty}^{-1} \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' , \\
 (II.2) \quad & F_2 = \rho_{\infty}^{-1} .
 \end{aligned}$$

We remark that the equation (1) when  $v = 0$  is exactly of the same form as the one for  $f_1$  we might have derived from the set of equations for the first-order approximations in the non-linear theory; however, these two solutions can be



easily obtained from the solution of  $p_1$  which is governed by a much simpler equation than  $f_1$ . From the knowledge of the reduced equation of (1) we can construct two integral equations

$$\bar{F}_1(\eta) = F_1(\eta) - v^2 \int_0^\eta K(\sigma, \eta) \bar{F}_1(\sigma) d\sigma, \quad (II.3)$$

$$\bar{F}_2(\eta) = F_2(\eta) - v^2 \int_0^\eta K(\sigma, \eta) \bar{F}_2(\sigma) d\sigma,$$

where

$$K(\sigma, \eta) = \frac{F_1(\eta)F_2(\sigma) - F_1(\sigma)F_2(\eta)}{F_1'(\sigma)F_2(\sigma) - F_2'(\sigma)F_1(\sigma)} \lambda^{-1}(u_\infty^2 - \lambda)$$

(II.4)

$$= -\lambda^{-1}u_\infty^2(\sigma) + \lambda^{-1}\rho_\infty(\sigma)\rho_\infty^{-1}(\eta)u_\infty^2(\sigma) + \rho_\infty^{-1}(\eta)u_\infty^2(\sigma)$$

$$\int_\eta^\sigma \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')} d\eta'.$$

For any  $\eta_0 > 0$  such that  $0 \leq \eta \leq \eta_0 < \infty$ ,  $0 \leq \sigma \leq \eta$ ,  $F_1(\eta)$ ,  $F_2(\eta)$  and  $K(\sigma, \eta)$  are bounded, by the theory of Volterra integral equations,  $\bar{F}_1(\eta)$  and  $\bar{F}_2(\eta)$  are uniquely determined by (II.5) and satisfy (II.1). As seen from (II.3) and (II.4),  $u_\infty^2 = \lambda$  is an apparent singularity of (II.1) for any bounded  $\eta$ . We also recall that the equation for  $\rho_\xi^*$  is

$$\rho_\xi^* = H(\xi)(u_\infty^2 - \lambda)^{-1}(\lambda\rho_\infty F - \rho_\infty u_\infty^2 F_\eta).$$



It follows from (II.3) and (II.4) that

$$\rho_{\xi}^* = H(\xi) \left\{ c_1 \left[ \int_{\eta}^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' + v^2 \rho_{\infty}(\eta) \int_0^{\eta} K_1(\sigma, \eta) \bar{F}_1(\sigma) d\sigma \right] \right. \\ \left. - c_2 \left[ 1 - v^2 \rho_{\infty}(\eta) \int_0^{\eta} K_1(\sigma, \eta) \bar{F}_2(\sigma) d\sigma \right] \right\},$$

where

$$F = c_1 \bar{F}_1 + c_2 \bar{F}_2$$

$$K_1(\sigma, \eta) = \lambda^{-1} \rho_{\infty}(\sigma) \rho_{\infty}^{-1}(\eta) u_{\infty}^2(\sigma) + \rho_{\infty}^{-1}(\eta) u_{\infty}^2(\sigma) \int_{\eta}^{\sigma} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta',$$

and  $c_1$ ,  $c_2$  are two arbitrary constants, and by  $F(0) = 0$ ,

$$c_1 \left( -\lambda^{-1} + \int_0^{\infty} \frac{\rho_{\infty}(\eta')}{u_{\infty}^2(\eta')} d\eta' \right) + c_2 = 0.$$

This shows that  $u_{\infty} = \lambda$  is also an apparent singularity of  $\rho_{\xi}^*$  for any bounded  $\eta$ .

At infinity, the equation (II.1) may not possess a bounded solution for  $F$  corresponding to a given bounded  $u_{\infty}(\eta)$ . In order to facilitate our discussion we introduce the variables:

$$G(\eta) = e^{-\eta} F(\eta),$$

$$z = e^{-\eta},$$

and (II.1) becomes

$$(II.5) \quad G_{ZZ} - (u_{\infty}^2 - \lambda)^{-1} (u_{\infty}^2)^{-1} \lambda (u_{\infty}^2)_Z G_Z = -v^2 \lambda^{-1} (u_{\infty}^2 - \lambda) z^{-2} G.$$



We assume that

$$u_{\infty}^2(z) = \sum_{n=0}^{\infty} b_n z^n$$

and let

$$G(z) = \sum_{n=0}^{\infty} a_n z^{c+n}.$$

Since the method of Frobenius is well-known, in what follows we only indicate the general results without going into details. There are several cases we need to consider:

$$(1) \quad b_0 = \lambda, \quad b_1 = b_2 = \cdots = b_{m-1} = 0, \quad b_m \neq 0.$$

In this case the two roots of the indicial equation are  $c = 0$  and  $c = m + 1$ . One solution of  $F$  is bounded and of  $O(e^{-m\eta})$  and the other contains a logarithmic term and is of  $O(e^{\eta})$ . It is shown that only the bounded solution satisfies the boundary condition at infinity.

$$(2) \quad b_0 \neq \lambda, \quad b_0 = \cdots = b_{m-1} = 0, \quad b_m \neq 0, \quad m \geq 0.$$

(a)  $b_0 \neq 0$ . The indicial equation is found as

$$c^2 - c + v^2 \left( \frac{b_0}{\lambda} - 1 \right) = 0,$$

and the two roots are

$$c \pm = \frac{1}{2} [1 \pm \omega],$$

$$\omega = \sqrt{1 - 4v^2 \frac{b_0 - \lambda}{\lambda}}.$$

We find two independent solutions for  $F$  free of logarithms



if  $\omega$  is not an integer. One solution is of  $O(e^{(\frac{1}{2} - \frac{\omega}{2})\eta})$ , the other, of  $O(e^{(\frac{1}{2} + \frac{\omega}{2})\eta})$ . If  $b_0 < \lambda$ , then  $\omega > 1$ , and only one of the solutions is bounded and satisfies the condition at infinity. If  $b_0 < \lambda$ ,  $\omega < 1$  and both solutions are unbounded.

For  $m > 0$ , the indicial equation becomes

$$c^2 + (m-1)c - v^2 = 0$$

and

$$c_{\pm} = \frac{1}{2} [ - (m-1) \pm \sqrt{(m-1)^2 + 4v^2} ] .$$

For small  $v$ , both solutions of  $F$  are unbounded.

Finally we note that all the terms we have neglected in the linearizing procedure, i.e.

$$f_{\eta}^* p_{\xi}^*, \quad f_{\xi}^* p_{\eta}^*, \quad G_{\infty} u^* f_{\xi}^*, \quad G_{\infty} u_{\infty}^* f_{\xi}^*, \quad \rho^* f_{\eta}^*, \quad \rho^* u^* f_{0\eta},$$

$$u^* \rho_0 f_{\eta}^*, \quad \rho^* f_{\eta}^* u_{\infty}, \quad \rho^* u_{\infty}^* f_{\eta}^*$$

are bounded if there exists a bounded solution for  $F$  at  $\eta = \infty$ . This justifies our linearizing procedure given in §3, Part III.



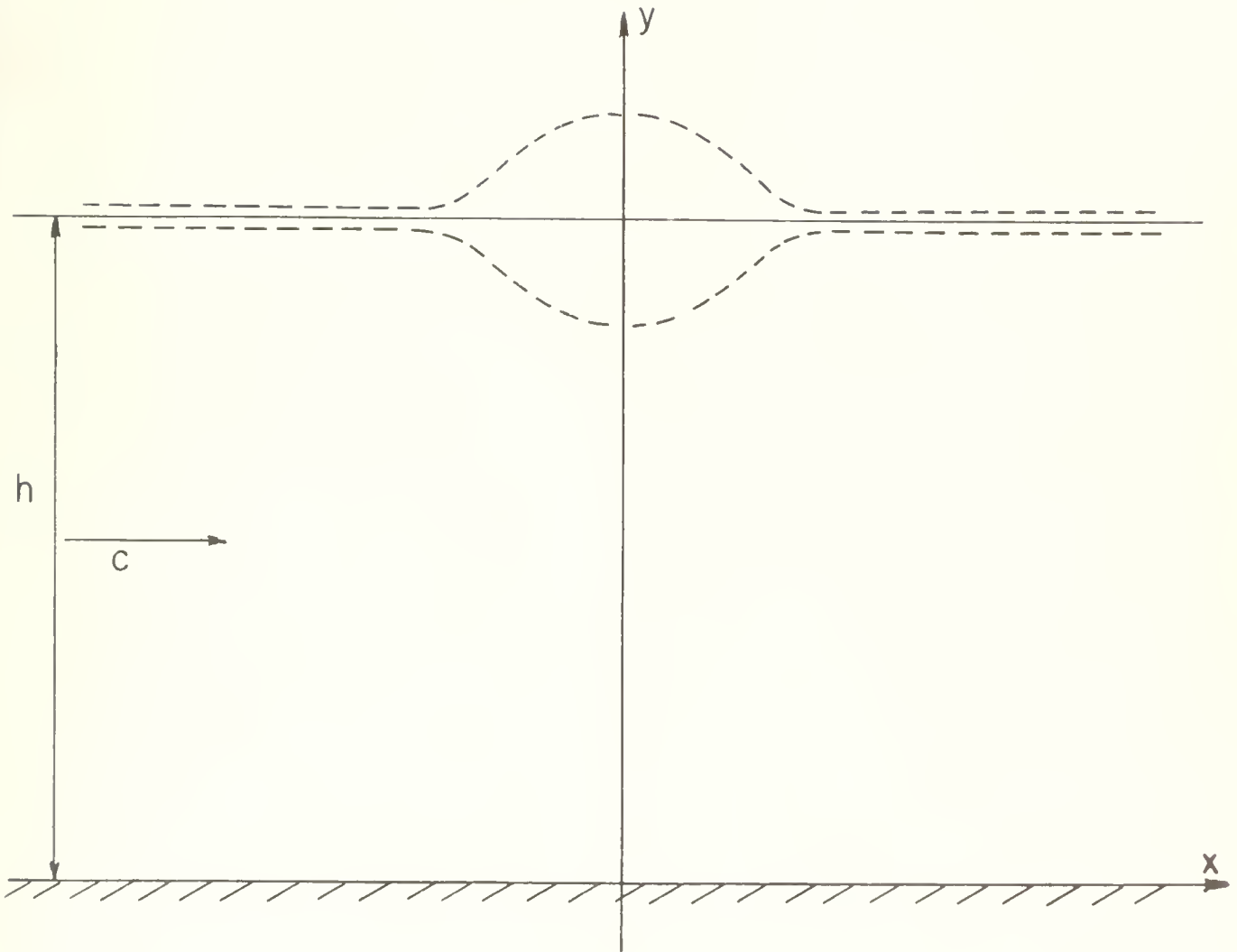


Figure 1



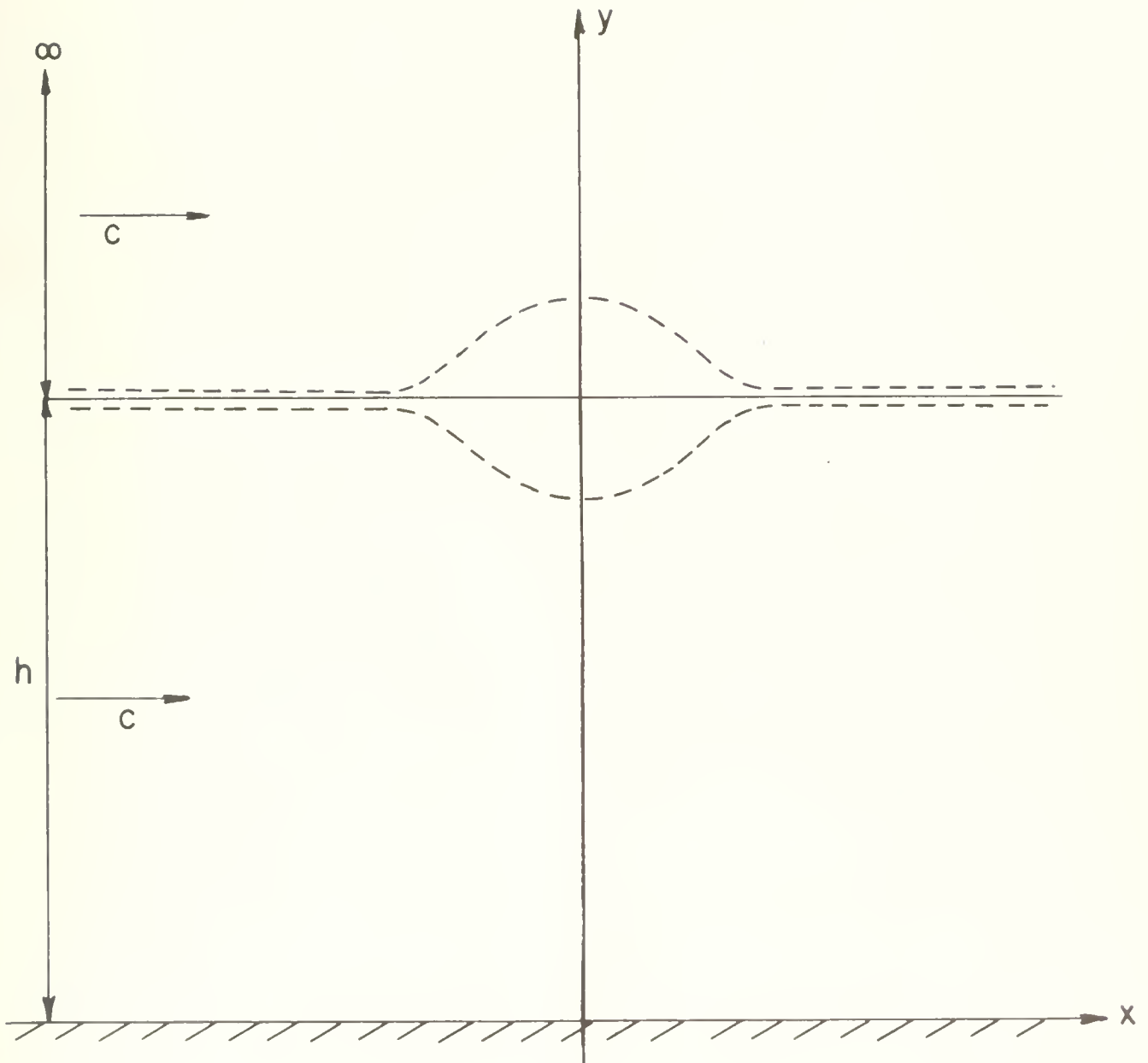


Figure 2



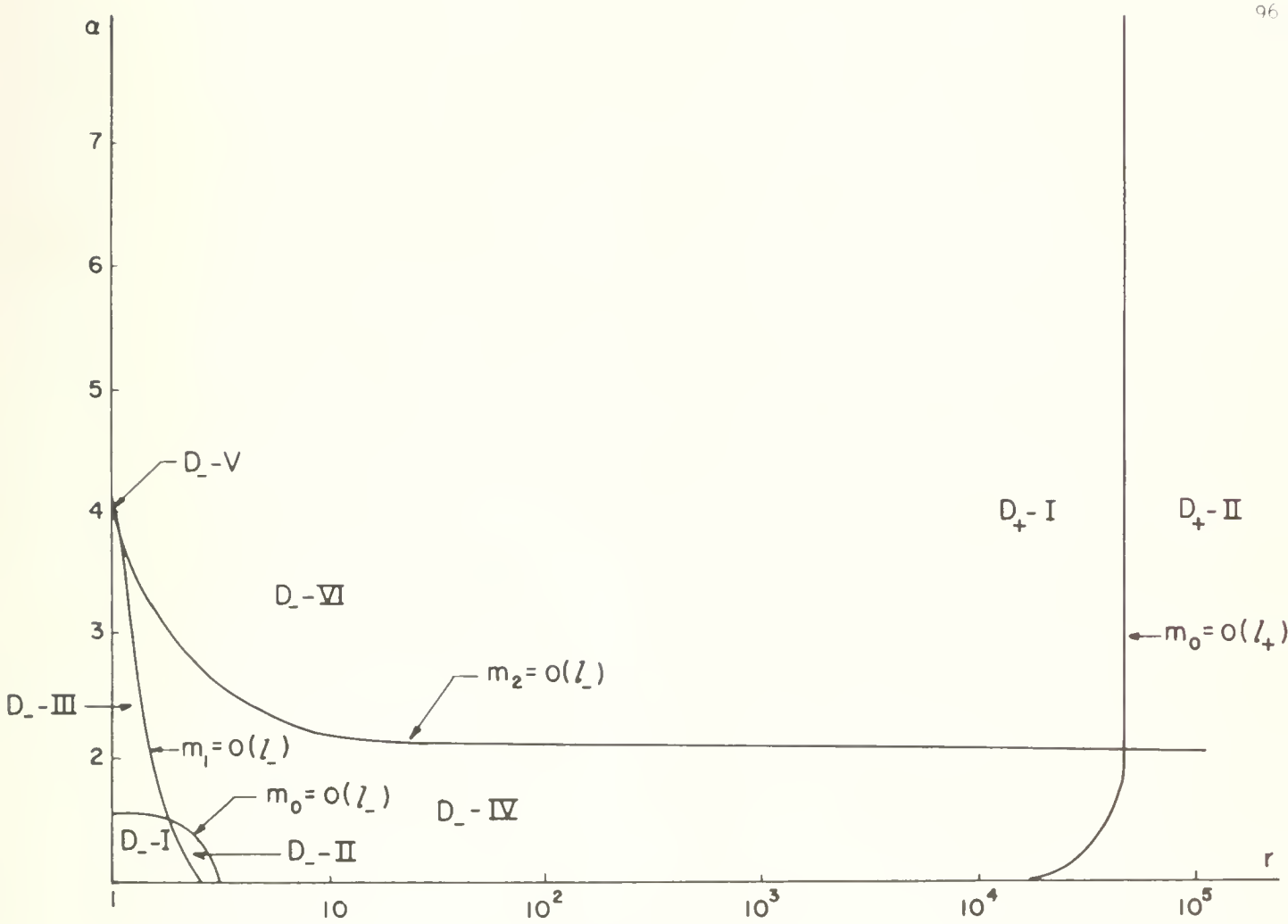


Figure 3



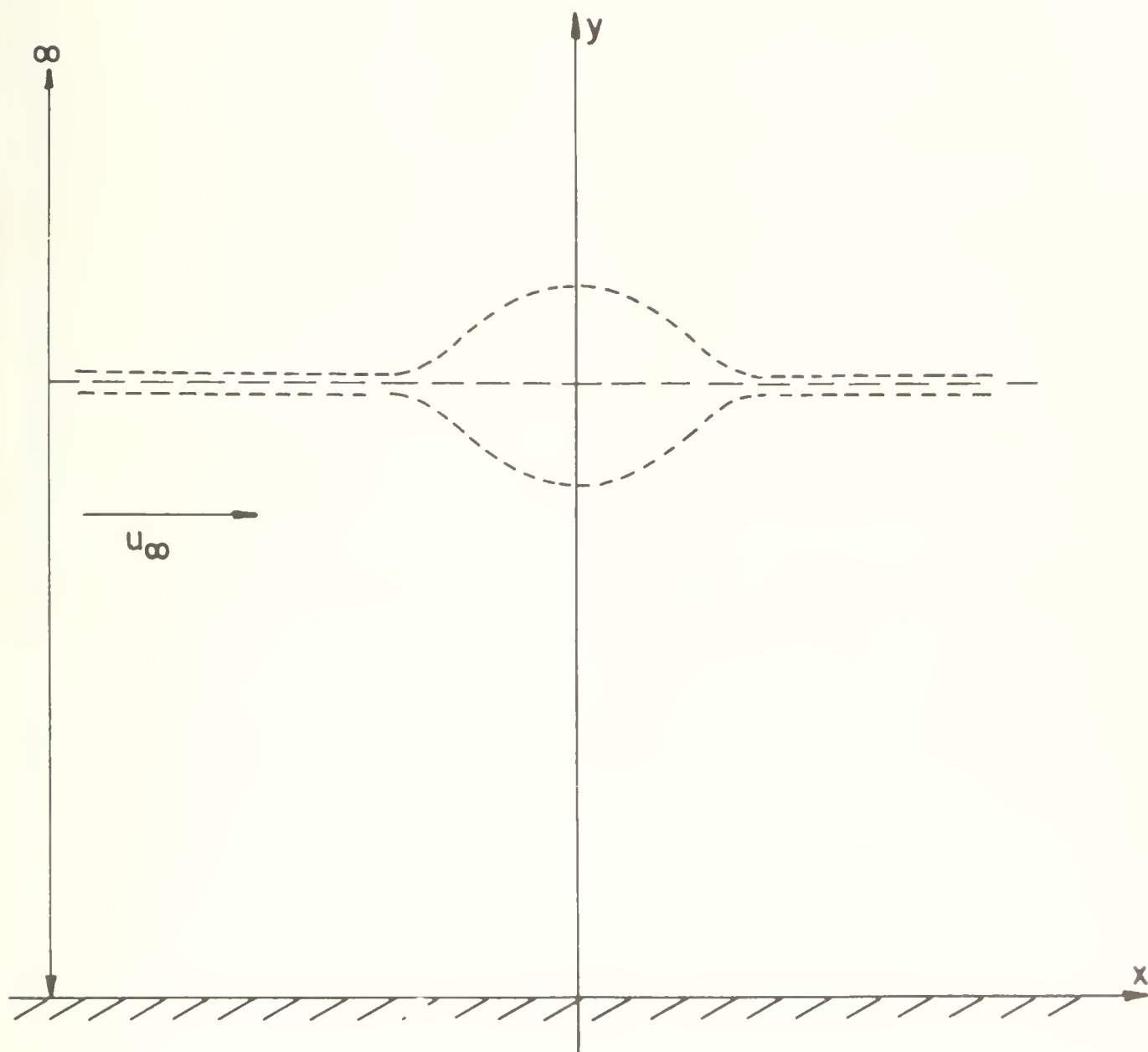


Figure 4



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